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Educação Matemática Debate

Fractions, show your true colors

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ARTICLE

Abstract: We present fundamental ideas for a proposal for teaching fractions. The theoretical foundation focuses on the critique of the overemphasized use of geometric models, which are commonly found but difficult to correspond into reality. In contrast, we find theoretical support in Realistic Mathematics Education and Freudenthal's developmental research. As part of the research procedures, we proposed to students realistic problem situations involving real whole objects and special parts of them, allowing students to develop solution strategies through reflection and discussion. We observed the emergence of meaningful concepts and strategies; fractions expressing quantities and being logically placed on the number line, with multiple representations for each; and informal understanding of the density of this set of numbers on the number line.

Keywords: Fractions. Representations. Reality. Numbers.

Fracción, muestra tu cara

Resumen: Presentamos una propuesta didáctica para la enseñanza de las fracciones basada en la investigación desarrollativa. La teoría critica el uso excesivo de modelos geométricos, comunes pero difíciles de aplicar en la realidad. En su lugar, nos apoyamos en la Educación Matemática Realista y la investigación de Freudenthal. Como parte de la investigación, propusimos a los estudiantes situaciones problemáticas realistas con objetos enteros y partes de ellos, permitiéndoles desarrollar estrategias de solución mediante reflexión y discusión. Como resultados, observamos conceptos y estrategias significativas; fracciones colocadas lógicamente en la recta numérica, expresando cantidades con múltiples representaciones; y la densidad de este conjunto de números en la recta.

Palabras clave: Fracciones. Plantillas. Realidad. Números.

Fração, mostra a tua cara

Resumo: Apresentamos ideias fundamentais de proposta didática para o ensino de frações baseada em pesquisa desenvolvimental. A fundamentação teórica faz crítica ao uso excessivo de modelos geométricos, com difícil transposição para a realidade. Em contrapartida, encontramos apoio teórico na Educação Matemática Realista e na pesquisa desenvolvimental de Freudenthal. Como parte dos procedimentos de pesquisa, propusemos aos alunos situações-problema realistas envolvendo objetos reais inteiros e partes especiais deles, e deixamos que desenvolvessem estratégias de solução, por meio de reflexões e discussões. Como resultados, vimos surgir o uso de conceitos e estratégias com significado; números fracionários expressando quantidades e sendo colocados de modo lógico na reta numérica, com múltiplas representações para cada um; e compreensão informal da densidade desse conjunto de números na reta.

Palavras-chave: Números. Fração. Realidade. Modelos.

"The failure of research-based instructional improvements to scale up successfully is one of the most vexing challenges facing educational research"
Catherine Lewis e Rebecca Perry (2017, p. 261)

1 Introduction

For one of us, the teaching and learning of fractions has been the focus of an ongoing research-study over the past thirty years (Bertoni, 2004, 2008, 2009). For the other, it is a source of frustration in the context of teacher education. The former's persistence can be attributed to several factors. One of them is the demand from public agencies and private companies for modules on the topic, as part of projects supporting elementary education. A second factor lies in successive and specific research interests, such as the historical dimension (Bertoni and Fiorentini, 1996; Bertoni, 2005), the construction of the concept of fractional number (Bertoni, 2008), and the articulation between part-whole fractions and the perception of an infinite set of numbers, quantifiers of collections made up of whole objects, or of parts derived from equal partitions, or both. A third factor is the emergence of new trends in the approach to the subject: the five meanings of fractional number (Behr *et al.*, 1983), unit coordination, connected fractional number sequences, and the iterative fractional scheme (Hackenberg, 2007; Olive & Steffe, 2001; Steffe, 2001), as well as the integrated teaching of fractions, decimals, and percentages (Lappan *et al.*, 2002, 2004; Lamon, 2012, 2017)[1].

We studied these trends, but they did not become the focus of our research, as we realized that what we sought was a strong and realistic formation of the concept of fraction, developed simultaneously with that of fractional number. In the five-meaning approach, however, it was already assumed that they were numbers. One of the meanings, the quotient, had already been part of our earlier research; the others seemed to us more like applications. As for the proposal to integrate the study of fractions with decimals and percentages, it seemed complicating and unnecessary for our purposes, since we had already developed solid knowledge about the topic and its didactics based on the part-whole conception, which we had been developing in a context that was, until then, quite distinct from the generic one.

But alongside these considerations, there is also a strong and frustrating subjective factor — the realization that, over these decades, little or nothing has changed in the teaching of fractions, and that students still face the same difficulties as before (Gabriel *et al.*, 2023; Hansen, Jordan e Rodrigues, 2017; Namkung e Fuchs, 2019; Singha *et al.*, 2021).

The second author was introduced to research in 1986 by the first author, who was then her undergraduate professor. Today, she holds a doctoral degree and, as part of her duties at Central Michigan University, teaches undergraduate students enrolled in a program that certifies them to teach from 3rd grade in Elementary School and 6th grade in Middle Schol. Feeling frustrated with her students' performance, the second author showed the first author some test results in which her students represented $\frac{1}{3}$ and $\frac{3}{8}$ using wholes of different sizes and *parts*[2] (thirds and eighths) that were roughly the same (Figure 1), or, in some cases, drew both the whole and the parts with no attention whatsoever to size (area or length), proportion, or congruence (Figure 2).

The first author's response came quickly: *"Poor model, Ana Lúcia. It works, partially, for those who already know. Descriptive and extremely limited"*. Ana Lúcia was surprised: *"A poor model? But isn't it present in all textbooks?* That's how a conversation and a renewed collaboration began, which gave rise to this article.

In textbooks (e.g., Dante and Viana, 2021; Silveira, 2021), we find the same abstract representations to introduce and support the development of the topic, the same abundant

[1] An expanded summary of this text was published in the proceedings of the *6th National Forum on Mathematics Curriculum*, held in October 2024.
[2] The term *parts* will be used to refer to the type of unitary fraction (halves, thirds, quarters, etc.), that is, the result of the equal division of the whole.

terminology, the same operational techniques that were present even before all the aforementioned research had been conducted. And more, students do not easily transfer relationships between the parts of a circle and parts found in real-life contexts (Pires, 2004).

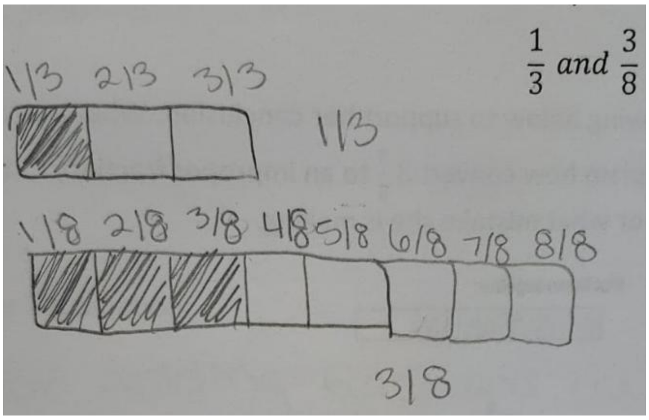

Figure 1: Representation with integers of different sizes and approximately equal parts (Own collection)

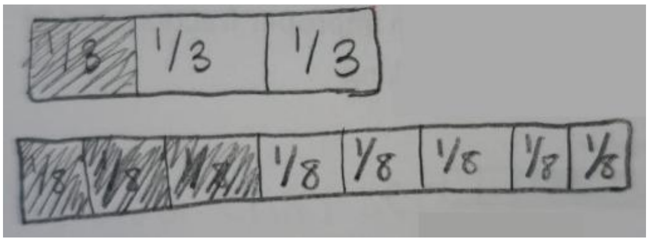

Figure 2: Representation with wholes and parts without considering size (Own collection)

We acknowledge that some textbook authors do include illustrations from real life — such as a plate with three quarters of a pizza — perhaps hoping that students will notice the relationship between partitioned plane figures and the real world. But these drawings act more like surface-level tokens or occasional reminders, leaving the impression that what truly matters is the school Mathematics content produced within geometric figures — that is, terminology, symbolism, operational techniques.

We also noted the presence of such representations in books by mathematics educators. Some books (Ripoll *et al.*) have intensified this imbalance: they present glimpses of fractions in real life, but then develop extensive treatment of the properties of geometric figures, going into aspects such as models, multiple equitable divisions — some unimaginable or even unthinkable in real life; and the reversibility of the relationship between units and the shapes/names/symbols of one or more equitable parts of the same whole. This unfolding ends up drowning out the presence of fractions in reality, which then appear only as a way to name parts of geometric figures.

With these considerations in mind, we set out to identify fractions as numbers, using equitable divisions of one or more real-world objects as a key resource. Our focus will be on the emergence of their capacity to express real-world quantities — an attribute that surfaces in the learning process and gives fractions their numerical character; on students' growing familiarity in recognizing numerical relationships among them, through mental abstraction rather than merely comparing the sizes of geometric pieces; and on understanding the infinite set of numbers they form — a set that extends the natural numbers and is dense on the number line. Furthermore, we aim to highlight the fact that division between any two numbers is always possible — that is, there always exists a number within this set that represents the result of such division. The topic of equivalent fractions will also be addressed, as it makes explicit that the fractional number is the common foundation shared by each of these classes. We believe it is essential for students to grasp this idea.

2 The problem: lack of reality and of numerical essence — oversized space for representations

Intermittently, studies on fractions in the curriculum of the early years of Basic Education have addressed two themes: whether and how we should teach fractions. Our position is the same as that expressed by Peter Hilton at the *IV International Congress on Mathematical Education* (ICME IV): that although we certainly should teach fractions as part of the elementary curriculum, in his view, we should not teach fractions the way they have been, and still are, taught (Hilton, 1983).

And what is that way? Suddenly, in the textbook, circles or rectangular strips divided into equal (congruent) parts appear — such as sectors of circles or smaller rectangles. Each whole figure, before being divided, is referred to as a *whole* or *entire unit*. This is the first time students encounter such a situation. The parts are given names and numerical signs. The names are assigned according to the number of parts into which the whole has been divided — giving rise to *a half*, *one third*, *one fourth*, *one fifth*, and so on. Those ending in *-ths* (in Portuguese, *avos*) appear at this stage or after the initial symbols. To present these latter, the activities propose that students paint some of the parts, whether adjacent or not; count how many were painted and how many there were in total; and write these results one above the other, separated by a horizontal line (the first result on top).

As one 4th grade student said when asked why he listed fractions as the most difficult topic in Mathematics:

> Because we have to do some stuff there, then we have to color, and when we color, then the rest — I don't really know. Because you color and then you have to do a bunch of numbers about the white part and the colored part. (In Santos, 2006, n.p.)

For example, if the circle is divided into two halves, the name *half* or *one-half* is assigned, as mentioned, along with the sign $\frac{1}{2}$, using the correct word to express it. The same process follows, in order, for dividing the figure into three, four, five, ..., eleven, twelve, etc., *equal* parts. These pedagogical proposals assume that, in this way, students will have learned the names and symbols for all *fractions* — terminology that begins to appear insidiously, referring both to the parts and to the numerical representation. A rather extensive terminology follows: *fraction, numerator, denominator, proper, improper, apparent, mixed*. All of these are supported by the aforementioned figures and the assigned signs.

That is, a fraction is considered *proper, apparent*, or *improper* depending on whether the number of colored or shaded parts is less than, equal to, or a multiple of, or greater than and not a multiple of the total number of parts in the whole. The category of apparent fractions leads to the somewhat hasty statement that every natural number is also a fraction. The case in which the number is greater than the total number of parts always causes some discomfort: how can one take more parts than those shown in the division of the circle?[3].

Indeed, this chapter of Mathematics, with its figurative, symbolic, and terminological apparatus confined to geometric figures, summarizes what Peter Hilton (1983) meant by "the way fractions are taught" (p. 37). It does not engage with reality. It does not seem to have any

[3] In a study of 4th graders, "although Joe indicated the hypothetical matchstick that would be ten times longer than the $\frac{14}{99}$ matchstick as $\frac{140}{99}$, he still indicated a disturbance when he asked, 'How can a fraction be larger than itself?'" (Olive & Steffe, 2001, p. 428). (Olive and Steffe, 2001, p. 428).

meaning in the students' daily lives.

It feels alien to the student: there was no motivating problem-situation — which, according to Vergnaud (1998), is a first necessary condition for learning — nor does it constitute something meaningful for the student, in the sense advocated by Freudenthal (1981). As a consequence, it also fails to reflect its quantifying nature in relation to specific collections from reality, which consist not only of discrete objects.

Therefore, in our view, many of the problems studied in Mathematics Education concerning the difficulties that children and student teachers have with fractions are created by the curriculum and the teaching methods. They appear as a need created within the school itself, not in the real world.

3 Methodological approach

It can be stated that the adopted methodology was that of a *long-term methodology process* (Figure 3), with alternation or simultaneity of classroom interventions, interactive teacher training, readings, reflections, and generated writings, in a continuous movement of increasing knowledge and observational applications, in the style of Freudenthal's (2002) developmental research.

Developmental research covers
longitudinal developments with a view on
long-term learning processes
and is itself a
long-term learning process.

Figure 3: Developmental research as a long-term learning process aimed at developing long-term learning processes (Freudenthal, 2002, p. 162)

What follows is a theoretical foundation for the methodology used. We will include narratives from some previously developed perspectives on the teaching and learning of fractions, as we consider them essential and integrative to our current broader view.

In 1994, Koeno Gravemeijer, from the Netherlands, wrote an influential article for the *Journal of Research in Mathematics Education* in which he described curriculum development and educational research in the Netherlands, where the concept of integrating design and research already had a long-standing tradition. Gravemeijer described curriculum development in that context as a kind of intentional and sensible improvisation, referred to as "theory-guided bricolage" (Gravemeijer, 1994, p. 443).

> In general, curricula are developed to change education, to introduce new content or new goals, or to teach the existing curriculum according to new insights. Notwithstanding the final goal of implementing change, development and implementation are completely separated in the conventional RDD model, which was in vogue in the '60s and '70s. In opposition to this approach, Freudenthal (1991) set out his concept of 'educational development.' This is more than just curriculum development; it also contains the end goal of changing educational practice. Educational development not only implies that the implementation of the curriculum is anticipated from the outset, it also implies that preservice and in-service teacher training, counseling, test development, and opinion shaping are incorporated in the development work. Starting in 1970, all of these activities were carried out by the Institute for Development of Mathematics Education (IOWO) under Freudenthal's directorship." (Gravemeijer, 1994, p. 445)

Gravemeijer points out that the term *educational development* was also used by Hemphill (1970) and Schutz (1970), but that in their view the concept did not include the research component, which was essential for Freudenthal. (Gravemeijer, 1994, p. 445)

> The feedback of practical experience into (new) though experiments induces an iteration of development and research. This cyclic process is at the center of Freudenthal's concept of developmental research. Unlike what is suggested by the RDD approach, development, practice depends on a cyclic alternation of development and research [...]. The cyclic process that Freudenthal discerns can also be seen as a learning process of the developer. Returning to the theory-guided bricolage, we may say that a global a priori theory which Freudenthal would prefer to call a philosophy guides developmental work. This theory functions as a basis for a learning process by the developer that is nurtured by the cyclic alternation of thought experiment and practical experiment. (Gravemeijer, 1994, pp. 449-450)

The aspect of time and the long duration of the cycles of thought and practice in developmental research are important to Gravemeijer and Freudenthal — and to the IOWO as a whole. Gravemeijer also draws on the concept of *bricolage* from Jacob (1977), "who refers to evolution as a kind of bricolage, for which he prefers the English expression 'tinkering'" (Gravemeijer, 1994, p. 447-448). We retrieved the quote from Jacob that Gravemeijer used in his original:

> Naturally, this takes a long time. Evolution behaves like a tinkerer who, during eons upon eons, would slowly modify his work, unceasingly retouching it, cutting here, lengthening there, seizing the opportunities to adapt it progressively to its new use. (Jacob, 1977, p. 1164)

Gravemeijer explains his analogy between developmental research and evolution:

> In developmental research, the evolutionary aspect is much more important, not in the sense of a random process channeled by natural selection, but as a goal-oriented process of improvement and adjustment: a processs that is guided by a theory that grows during the process. (Gravemeijer, 1994, p. 451)

It is also important that the justification of *results* — in Gravemeijer's terms, discoveries — in this type of research departs from the positivist approach, which still heavily influences educational research. It is a justification based on arguments, not on the reproduction of empirical data.

> In the case of developmental research, theory is not put to the test after the development has been concluded. Instead, it is the developmental process itself that has to underpin the theory. In the cyclic process of development and research as sketched by Freudenthal, discovery and justification are closely interwoven. Discovery is not restricted to the thought experiment, and justification is not merely found in the results of the trials; some discoveries are made in the trials phase, and part of the justification is not empirical. Justification is also found in the thought experiment; however, it is then justification based on arguments. In a positivist interpretation, justification is confined to empirical testing; in the case of developmental research, however, the rationale for one's choices and the interpretation of the empirical data are

> part of the justification as well. This is connected with a shift from what Habermas (1987) calls “Zweckrationalitit” (means-end rationality) to what he calls “kommunikativer Rationalitat” (communicative rationality). The positivist rationality, which only takes into account means-end relations, is exchanged for a broader kind of rationality based on argumentation and comprehension.” (Gravemeijer, 1994, p. 453)

The developmental research, within which our discussion in this article is situated, involves the alternation of classroom interventions, interactive teacher training, readings, reflections, and the production of curriculum materials (booklets and modules) for teachers and students.

It began within the scope of the project *A new Mathematics curriculum from 1st to 8th grade — subprogram for the Teaching of Science – SPEC – MAT – UnB/MEC/CAPES/PADCT2*, coordinated by the first author. The project integrated research on curricula, experimentation with diverse groups of students, dissemination to teachers, and implementation in public schools in the Federal District. The project took place from 1985 to 1989 and had five lines of action:

> a) the definition of socially relevant Mathematics topics and their suitability to the interests and cognition of students in [the former] primary education (1º grau); b) in-depth study of the content and methodology of these topics, with the development of proposals for teaching in [the former] primary education; c) experimental activities to apply the proposals developed in item b, in two settings: at LEM (Mathematics Education Laboratory at UnB) with children, and in primary school classrooms in a pilot school; d) dissemination of these proposals and experiences to teachers through meetings, seminars, and newsletters; e) theoretical research on Mathematics Education and curricula in Brazil and other countries. A total of 3,571 public school teachers were reached, which at the time represented 45% of the educators involved in the practice of Mathematics teaching at the Department of Education of the Federal District (Muniz *et al*., 2009, p. 8).

What remained from it, in relation to fractions, were questions such as: To what extent do arithmetic relationships and operations involving fractions require graphic illustrative support? How can the logic of operations be made explicit to ten-year-old children?

The next phase of this developmental research, which constitutes long-term learning for the developers, was the *Gestar I* aimed at training Middle School teachers, in which we participated as authors of modules and as consultants responsible for reviewing all the modules. Another phase of the project with the same goal was the *Pedagogy Course for In-Service Teachers* (PIE), a partnership between the State Secretariat of Education of the Federal District and the Faculty of Education of the University of Brasília, from 1999 to 2005.

From 1997 to 1999, there was the *Pro-Science* project, a partnership between Brazilian Federal Agency for Support and Evaluation of Graduate Education and Open University of the Federal District (UNAB), in which the first author coordinated the Mathematics course and wrote modules, and the second author also wrote modules. Each of these stages fed into the next, contributing to our current view on the teaching and curriculum of fractions.

In the 2000s, more specifically from 2001 to 2007, the Ministry of Education launched the *Gestar II* program, which involved the creation of 24 modules, eight of which we authored. In this project, we also participated in the training of regional teacher educators in the North, Northeast, and Central-West regions of Brazil.

4 Lamentations — a literature review

In the wake of the predominant geometric models used in fraction teaching, research has also centered around them. The literature typically classifies the models used to visually represent fractions into two types: *area models* and *linear models*. The most well-known linear model is the number line. Area models represent fractions as parts of geometric figures, such as circles or rectangles. Usually, a polygon or a circle is used to represent the unit, and it is divided into congruent parts. The denominators of the fractions are represented by the total number of congruent parts into which the figure is divided, and the numerators are represented by the number of shaded parts.

Teaching fractions using area models, which, in the view of the student interviewed by Santos (2006), is "doing a bunch of numbers about the white part and the painted part" (n.p.), merely trains the student in a double-counting process that has little to do with the concept of fraction as a quantity or as a number.

> Clearly, double counting is a process that leads to the development of fraction language based on enumeration of parts, but not to the construction of the concept of fraction in its various aspects; that is, the child does not perceive this symbolic representation either as a fractional number or as a representation of a quantity. (Campos *et al*., 1999, p. 15)

As early as 1986, Daphne Kerslake, after having developed a six-session teaching sequence of 40 minutes each, tested by several teachers with their own classes, concluded that "[m]ost importantly, it seems that the dependence on diagrams inhibits the appreciation of the idea that fractions are numbers" (Kerslake, 1986, p. 97).

Even so, many more recent studies are still being conducted to determine which graphical representation of fractions is most effective. And textbooks continue to introduce fractions using such diagrams. A sad example of research not being applied to educational reality. Among these more recent studies on fraction representations, we find the experiment by Hamdan and Gunderson (2017), in which the researchers examined the performance of 2nd and 3rd grades students on fraction tasks after three types of treatment: number line training, area model training, and non-numerical training — the control group completed crossword puzzles. We note the use of the word *training* in all the experimental studies mentioned in this literature review, which may render the results invalid for the classroom, where, we hope, there is no training, but rather education. The training activities given to the participants in the study are shown in Figures 4 and 5.

Two things stand out to us in these training sessions. The text "fractions have a top part and a bottom part, like this number, $\frac{1}{4}$. This is a fraction. It has a number on top and a number on the bottom" is quite problematic, both from a mathematical and didactic point of view. First, because it refers to two numbers (parts of the representation), rather than to a new number indicated by this representation. After all, a fraction is not the numerical representation itself. Second, because they aim to give an example of "a number on top and a number on the bottom", but they write 1/4 with the numerator and denominator at the same height.

The number line was presented already completed, not constructed by the students. This is not surprising in a training context. But the fact that the line was actually a rectangle is puzzling. The authors justify this choice by stating:

> We used a slightly two-dimensional number line during training to avoid a common error children make with ruler measurement: counting hatch marks

as objects instead of counting the spaces between hatch marks (e.g., Solomon, Vasilyeva, Huttenlocher, & Levine, 2015). We expected that using a thin rectangle would guide children's attention to the spaces between hatch marks as the meaningful units. (Hamdan and Gunderson, 2017, p. 589)

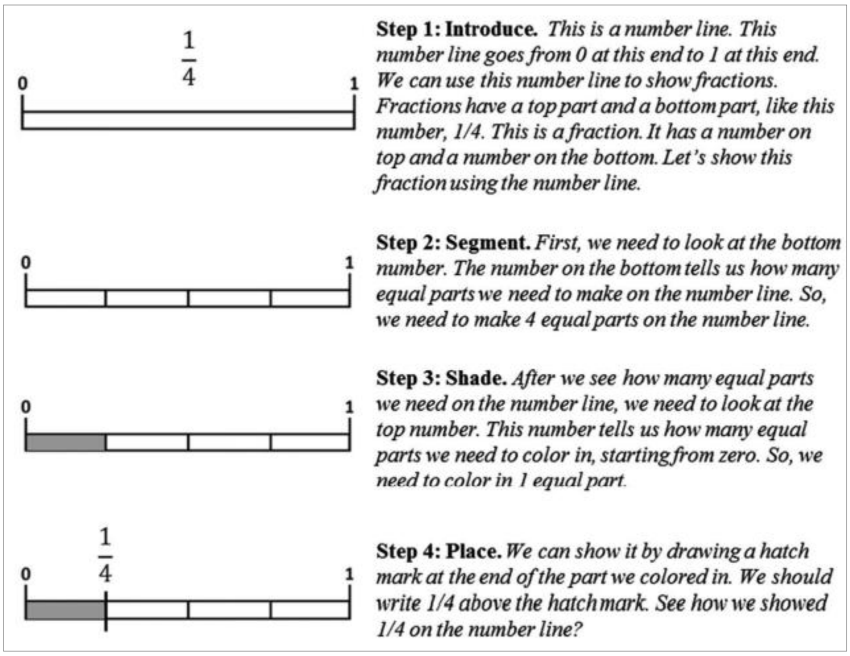

Figure 4: Number line training procedure (Hamdan and Gunderson, 2017, p. 590)

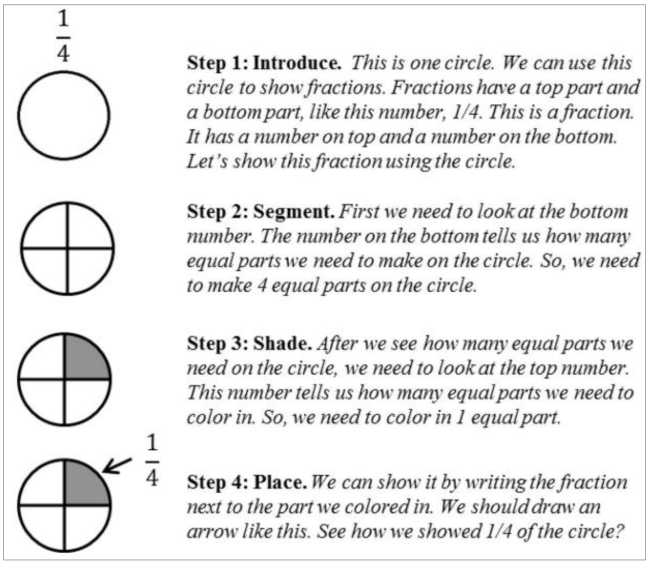

Figure 5: Area model training procedure (Hamdan and Gunderson, 2017, p. 590)

In our view, the choice demonstrates how models (in this case, rectangles) become *indispensable* and inflexible. The results indicated that only the number line training led to transfer in a fraction magnitude comparison task that differed from those offered in the treatment. The authors concluded that the number line plays a causal role in children's understanding of fraction magnitude and is more beneficial than the area model.

Later, the same authors joined other researchers and conducted a new experiment (Gunderson *et al*., 2019). This time, This time, 148 2nd or 3rd gardes students were randomly assigned to four groups, each receiving a different type of training using what the researchers called a pure unidimensional number line, a hybrid unidimensional number line, a square number line, and an area model (Figure 6).

The treatments consisted of 15-minute training sessions conducted by the experimenters. Among other results, the pure unidimensional number line outperformed the area model training in fraction magnitude comparisons. The researchers concluded: "We argue that unidimensionality is a critical feature of the number line" (Gunderson *et al*., 2019).

It is interesting that they argue this, given that a line is, mathematically, always unidimensional — and the artifacts created by the researchers should never have been called lines.

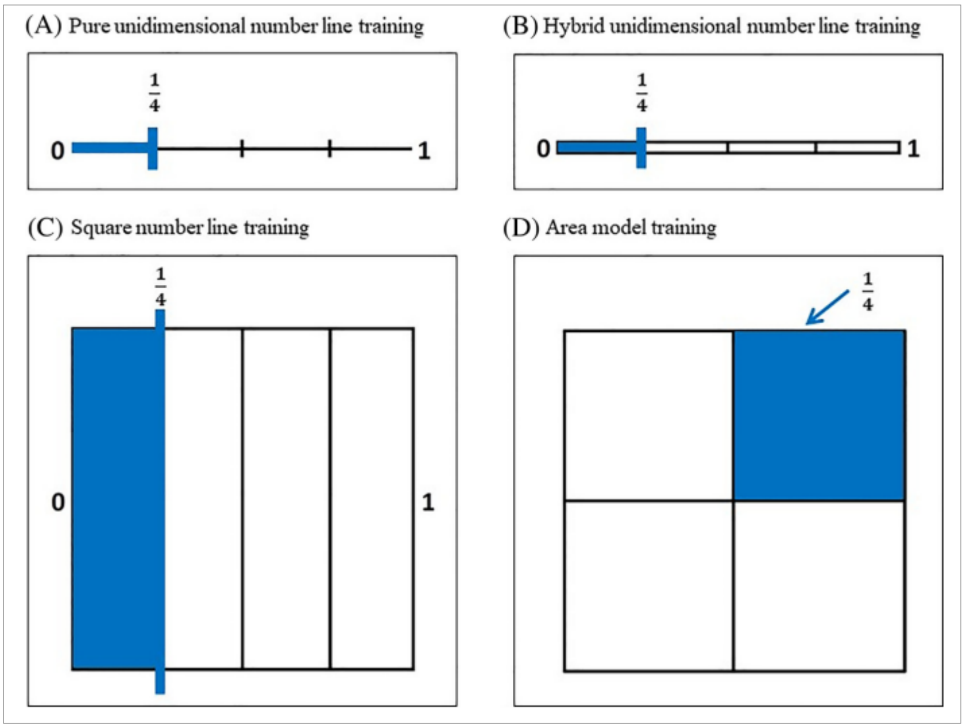

Figure 6: Example of the representations used during training in each condition.
(Gunderson *et al*., 2019, p. 4)

A supposed superiority of the linear model in experiments was also the result of a study conducted by Sidney, Thompson, and Rivera (2019) with 123 5th and 6th grades students, who solved fraction division problems after receiving one of four types of training: number lines, circular area models, rectangular area models, or no visual model. The children who solved the problems accompanied by a number line were more accurate and showed evidence of consistently producing solid conceptual models in most problems compared to the children who solved the problems with any area model or with no visual model at all (Sidney, Thompson and Rivera, 2019). The authors found these findings particularly surprising because circles and rectangles were likely familiar to the children, given that students encounter models of area division into equal parts as early as first grade.

Tian et al. (2021), however, found different results. They randomly assigned 4th and 5th grade students to one of three groups: one that received number line training, one that received area model training, and a control group that received non-numerical training. The researchers reported that training with the area model led to improvements in children's performance on estimation and fraction magnitude comparison tasks. However, contrary to their expectations, number line training did not improve children's performance on that type of task, and none of the training types led to improvements in transfer tasks that assess understanding of fraction magnitude.

As we have already mentioned, the results of these studies should be interpreted with caution. They were conducted in clinical experimental contexts, which follow strict procedures and do not allow for more open interactions with students or exploratory activities. We contend that, if the experimental methods of the natural sciences were truly suitable for educational research — and if Hamdan and Gunderson's (2017) conclusion that number line training causes success in comparing fraction magnitudes were valid — then the participants in Tian et al.'s (2021) study should have succeeded in the fraction comparison tasks after being *trained* with the number line.

Cramer, Post, and Delmas (2002) also used an experimental research design. However, the *treatment* was not limited to brief training sessions with only one type of fraction representation. They compared the performance of students using commercial curricula (CC) for initial fraction learning with the performance of students who used the Rational Number Project (RNP) fraction curriculum. The RNP curriculum placed particular emphasis on the use of multiple physical models and on the transition between different modes of representation — pictorial, manipulative, verbal, real-world, and symbolic.

The instructional program lasted 28–30 days and involved more than 1,600 1st and 4th grade students across 66 classrooms, which were randomly assigned to treatment groups. The quantitative part of the research (pre- and post-tests) showed a significant difference in performance between the two groups, with the RNP group performing better. The qualitative part of the research (interview data) revealed differences in the quality of students' thinking when solving ordering and estimation tasks involving fractions. RNP students approached the tasks conceptually, building their reasoning from mental images of fractions, while CC students more frequently relied on standard, often mechanical, procedures when solving fraction tasks.

> Though the circle model was the primary manipulative in this study, concerns that this model reinforced whole number thinking did not materialize. Such problems may occur when students are limited to using a single model and do not examine the model while the unit is continually and systematically varied as in this study. (Cramer, Post and Delmas, 2002, p. 119)

All the studies we found specifically about fraction representations were of the experimental type. It is true that many were published in psychology journals. But the study by Cramer, Post, and Delmas (2002) was published in the *Journal of Research in Mathematics Education*, the most prestigious journal in Mathematics Education in the United States, with an acceptance rate of approximately 6% of submitted manuscripts (Langrall, 2015). We hope that this literature review has sparked in the reader the same indignation we felt regarding the type of studies that have been published on the issue we intend to discuss in this article. And it is precisely this type of research, labeled by the U.S. Department of Education as *scientifically-based research* — narrowing the definition of science to experimental studies — that influences educational policies, curriculum guidelines, and textbooks (Lather and Moss, 2005).

Allow us a touch of affectation: there should one day be a public demonstration of recognition of error, with an apology from authors of Mathematics textbooks and mathematics educators for the violence and waste of time inflicted on students.

Considering the differing results obtained by Sidney, Thompson, and Rivera (2019), who found limitations in the use of area models, and those by Cramer, Post, and Delmas (2002), who did not find the same issue, we believe there were reasons for this difference. The latter achieved positive results among students who used the fraction curriculum of the *Rational Number Project* (RNP), which places particular emphasis on the use of multiple physical models and on the transition between modes of representation — pictorial, manipulative, verbal, real-world, and symbolic — going beyond the restrictive use of area models and broadening the spectrum of approaches to the topic. This inference from the research supports the choices we made, which went even further, to the point of virtually eliminating the use of geometric area models.

Regarding the theme of fractions on the number line, we observed some tentative efforts in the study of number line representation and the placement of fractions on it. We detected difficulties in establishing a unit and confusion regarding the linear essence of the number line. We noticed a kind of evasion from this essence, as seen in Figure 6. In it, the mathematical, linear line is referred to as the *pure unidimensional number line*. They also present a *hybrid unidimensional number line*. The two other models also rely on two-dimensional figures, but this one seems to be irrefutable proof of the enormous difficulty in introducing any new concept or fact into the theory without resorting to rectangles — when rectangles have been dominant in the development of the topic. Despite these difficulties, there were claims that the linear model facilitated the recognition of fraction magnitudes. Our reflections are that, in our research, we did not encounter such difficulties. The linear number line was already familiar to students for marking quantities of whole objects, and since they had encountered parts of those objects in real life, adding those part markings came naturally. The *magnitude* of fractions had already been acquired through the recognition that they expressed quantities — an idea inherent to number sense and the perception of order.

On the other hand, our goal went beyond merely placing fractions on the number line. It aimed at enabling the perception of the final result: the distribution of fractions all along the number line, so close to each other, and still making it possible to perceive that another one could be placed between any two. The visual imagination led to the anticipation that they would fill the number line. We achieved this goal, and we also called attention to the fact that many other numbers could fit on the number line — *even more* than just the fractions. However, up to now, we have not found studies that contribute in that direction.

In search of research focusing on fractions in the real world, we found Mack (1990; 1995), who studied the influence of what she called *informal knowledge* of fractions on the acquisition of formal symbolism and procedures. In 1990, Mack conducted a study with eight sixth-grade students. These students had studied fractions in fifth grade using a traditional textbook approach. During the research, the students received instruction that focused more on the conceptual aspect of fractions rather than on procedures. The questions were asked verbally. Concrete materials, pencils, and paper were made available to participants, but their use was not encouraged.

As results, Mack reported that all eight students began the study with incorrect ideas regarding the use of symbolic notation and fraction procedures; on the other hand, they also had substantial informal knowledge of fractions, which enabled them to solve numerous problems presented in real-world contexts. This informal knowledge was shown to be disconnected from their knowledge of fractional notation, procedures, and concrete representations. Mack also observed that participants appeared to suffer interference from memorized procedures when attempting to solve problems — whether the problems were presented symbolically or in real-life situations.

She concluded that students can build symbolic knowledge and procedures based on their informal knowledge only if they are able to relate the two. In 1995, Mack suggested that informal knowledge might arise, among other sources, from everyday situations experienced with family members. In our view, however, the knowledge demonstrated by the students seemed to come not only from everyday situations, but also from a learning process consistent with schooling, since the students showed proficiency in an initial phase of partitioning and reasoning — beyond what is typically developed in the family environment — reaching the point of informally solving problems such as $4\frac{1}{8} - \frac{7}{8}$, posed in word problems.

This is a type of knowledge we consider highly relevant to mathematical learning, as it constitutes a grasp of the roots of ideas — whether historical or expert-generated — about a new domain. This, before the creators provide the formal apparatus to store that knowledge — although they often use it, inappropriately, to present it. Yet, although the students had already been introduced to symbolic notation, they did not show the same level of performance with it. Our interpretive hypothesis is that the core of the research should not have been to reveal the power of the non-formalized in acquiring the formal, but rather the didactic adequacy of how the formal is presented so that it can be merged with the former. Mack's (1995) research was highly valuable to ours, in that it linked symbolic knowledge with life situations—not with graphical representations in geometric figures.

5 Transition to Real-World Collections

Several issues encountered throughout the first author's journey called for alternative paths. One of them was the realization that the topic is often introduced through the use of circles or rectangular bars divided into equal (Dante and Viana, 2021; Silveira, 2021), usually congruent, parts. As we have stated, this is something unfamiliar to the student. But, knowing that the purpose is to explore numerically special parts of objects — whether together or not with some whole ones, we then asked: Why not draw directly from reality, where such collections are abundant? Why work with representative schemes and not with what is being represented? This option was recorded by the first author in Bertoni (n.d.), and we highlight a few points here.

6 Discoveries (also known as results)

During the work in the project *A new Mathematics curriculum from 1st to 8th grade*, and in working with children for the development of the proposal *Play, Think, Do [Brincar, Pensar, Fazer]* (Bertoni, 1994), we proposed realistic problem situations involving real whole objects and special parts of them, and allowed the children to develop spontaneous solution strategies through conversations and discussions. We observed the emergence of concepts and strategies with meaning. The strategies were often non-conventional.

The *half* or *one-half* part is socially known and readily recognized by children. Most of the time, it appears alongside whole objects — what textbooks refer to as mixed fractions: one and a half sandwiches, two and a half oranges, one and a half laps, two and a half hours, three and a half blocks. This reality that we adopted naturally includes more than two halves, without anything to justify the textbook-assigned label of *improper*. In the real world (and not in the restricted fractional world of didacticians), halves appear in numbers far greater than two. On a journey of one hundred and a half kilometers, for example, if we were to count the half-kilometers traveled, we would have 201.

Likewise, other sets or expressions emerge in everyday life involving whole objects along with special parts of them: fourths, tenths, hundredths. In our work, halves led students to think about *halves of halves*, which we explained are called fourths, because they involve dividing the object by four. Then came halves of fourths, which resulted, for instance, in pizzas divided into eight parts, each called one eighth.

With the concept of object division already accepted and made meaningful, we introduced other cases, grounded in engaging situations involving food sharing — like dividing by three, yielding thirds, and then through further subdivisions, sixths and ninths. In a similar way, we built up fifths and tenths. At each stage, various relationships were established — some simple, others more complex. The relationship of one-half being equal to two fourths spread like wildfire. Students knew mentally, for instance, that one fourth plus one fourth made a half, that a half minus a fourth gave one fourth, and even that a half plus a fourth gave three fourths.

6.1 The emergence of the fraction as a quotient

The halves of halves brought us other surprises. When the students had to divide 3 sandwiches among the four of them, they first split them in half, resulting in 1 and a half sandwiches, then split again, obtaining a half and another quarter. They knew the value of that amount, three quarters, and we drew attention to that: 3 divided by 4 equals 3 quarters. We were familiar with the more common or canonical situation to obtain this result, with a different focus, and took the opportunity to present it: 4 young people went to a pizzeria and ordered 3 pizzas. The waiter immediately divided all 3 into four parts. Each person took one part of the first, then one part of the second, and one of the third. How much did each one eat? Also in this situation, we have 3 divided by 4 equaling 3 quarters. However, there was no reaction among the students. It seemed they preferred to think in terms of halves of halves. These are opportunities that spark ideas and surprise us.

We open a parenthesis here to share an illustrative moment of the *long-term learning process* experienced by the researchers in this developmental research. For the first author, it was essential to include the record of the division of 3 by 4, carried out by the students, in which they obtained the result $\frac{1}{2}+\frac{1}{4}$, quickly recognized as $\frac{3}{4}$. For the second author, who is also a historian of Mathematics, it was striking that this process was analogous to the one used by the ancient Egyptians—and that it is always possible. She pointed out that in ancient Egypt, what most resembled our concept of fractions involved only parts, or unit fractions (with two exceptions: two-thirds and three-quarters). Many of us today find this remarkable. We ask why. Why didn't they use, for example, the fraction $\frac{5}{7}$? Well, to them, that was a division: five divided by seven (Bunt, Jones and Bedient, 1988). And they did not express the result as five sevenths — just as the children in the previously described situation did not like giving the answer as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$). They required all the parts used (all types of unit fractions) to be different — perhaps aiming to present unit pieces as large as possible. One of the results the ancient Egyptians gave for that division was *half plus one-seventh plus one-fourteenth* (see Figure 7).

Figure 7: Two possible results of the division of five by seven (Own elaboration)

This shared insight by the authors resulted in the creation of an activity to be proposed, during each author's future interactions with learners, in upcoming sessions, with observations on the interest and performance that may arise:

Activity. In an ancient country, a man had to divide four drex coins equally among five servants. Coins representing unit fractions of a drex existed and were available: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$*... of a drex. Would it be possible to make the payment using only these coins, in such a way that each person would receive no repeated coins?*

1) Carry out this division. (One idea is to test, step by step, whether the man could give $\frac{1}{2}$ to each person, then see whether he could still give $\frac{1}{3}$ to each, and so on until the 4 drex coins are used.)

Solution: Giving ½ to each person would require five $\frac{1}{2}$ coins, and he would still have $1\frac{1}{2}$ drex left. He cannot give five $\frac{1}{3}$ coins. But he can give five $\frac{1}{4}$ coins, leaving $\frac{1}{4}$ of a drex remaining. That last $\frac{1}{4}$, divided among 5, would be $\frac{1}{20}$ each. So each person would receive: $\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$.

Without requiring the use of the largest possible unit coins, the problem has another solution. It could start the same way, by giving $\frac{1}{2}$ drex to each person. This would leave 1 drex and $\frac{1}{2}$ drex. It is likely that students will realize both remaining amounts can be divided by five, yielding unit fraction coins. In fact, 1 divided among 5 gives $\frac{1}{5}$ for each, and $\frac{1}{2}$ divided among 5 gives $\frac{1}{10}$ for each. In total, each person receives: $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}$ drex.

2) *Express the total amount of drex each person received. Was this result expected?*

This narrative also illustrates the cyclical nature of this category of research.

But how can we go beyond this intuitive process that led both children and the ancient Egyptians to construct fractions as results of division?

Even without this historical application, real-life situations requiring numerical results involving fractions inevitably lead to the formation of the concept of fraction as a number. Our proposal is that, starting from part-fractions, with their part-whole relational character, there should be a progressive elaboration of their numerical nature — expressing quantities — which will lead to the construction of an infinite set of numbers, densely distributed along the number line, alongside natural numbers. Parts of objects, whether isolated or combined with wholes, will serve as a foundation for fractional numbers, or positive rational numbers.

6.2 Placement of fractional numbers on the number line. Equivalent fractions corresponding to a single number. The density of fractions on the line.

As suggested by Bertoni (n.d.), we proposed the gradual placement of other fractional numbers on the number line, which had already been marked with the natural numbers. For younger children, the concepts of distance or measurement seemed strange in the narrow context of small segments. They couldn't perceive distance in those tiny pieces. But they could see analogies with the placement of natural numbers on the line, primarily through the quantities expressed by each. With only the natural numbers drawn on a number line on the board, and saying we were going to count oranges, we placed small posters along the sequence with illustrations of one orange, two oranges, etc. Then we presented a similar poster showing half an orange and asked: *"Here we have half an orange. Where should we place this poster and write the number $\frac{1}{2}$?"* Most students responded promptly, suggesting the position halfway between 0 and 1.

In a way, we used a more positional language, such as: *"If from here (zero) to here equals 1, where do we place a point to mean (or represent) a half?"*. Thus, the first numbers to be placed were those of the type $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, *etc*.. Note that two halves, four halves, etc., coincide with the natural numbers. We placed them just above the previous marks for the natural numbers. Also, above the mark for $1\frac{1}{2}$ we placed the mark $\frac{3}{2}$; above $2\frac{1}{2}$ we placed $\frac{5}{2}$, and so on. Then we could mark the fourths: $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $1\frac{1}{4}$, $2\frac{1}{4}$, etc. In our approach, vertical stacks of overlapping numerical representations begin to appear (Figure 8). All of them correspond to a single point on the number line, all at the same distance from zero, all representing quantities that are equivalent — therefore all corresponding to a single number.

Students begin to build the understanding that a fractional number is like this: it has countless different representations. Another thing they notice is how the number line starts to fill up with these new numbers. We begin by focusing on the interval from 0 to 1. After placing the halves $\left(\frac{1}{2}\text{ and }\frac{2}{2}\right)$ and fourths, we can mark the thirds and sixths. The number $\frac{1}{3}$ is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. It is placed between $\frac{1}{4}$ and $\frac{1}{2}$. One sixth lies halfway between 0 and $\frac{1}{3}$; $\frac{3}{6}$ coincides with $\frac{1}{2}$. And $\frac{1}{5}$ is smaller than $\frac{1}{4}$. One fifth and $\frac{2}{5}$ are placed before $\frac{1}{2}$; $\frac{3}{5}$ and $\frac{4}{5}$ come after. One tenth $\left(\frac{1}{10}\right)$ is marked as the halfway point between 0 and $\frac{1}{5}$. Marking all the tenths from 0 to 1 already becomes a bit difficult. The children decide to draw another number line with wider intervals between the natural numbers. Next, we mark the twentieths — each one being half of a tenth.

And then we'll have to mark the thirtieths, ... hundredths, thousandths... For the last one, we'll need to divide the interval from 0 to 1 into 1,000 parts. That's not so impossible: if we take the unit as one meter, it already appears divided into thousandths. And we can divide it even further: into millions, trillions of parts... At that point, students begin to think that the number line will become completely filled with fractional numbers, and that it won't even be possible to separate one point from another. That's the moment to *reveal the hidden twist* — to say that there will still be plenty of space left, more than what is occupied by the rational numbers (Mathematics has its mysteries). And that they'll learn about this once they study decimal numbers. They'll discover that there are infinitely many of them that are not fractional numbers, but that must also have their places on the number line.

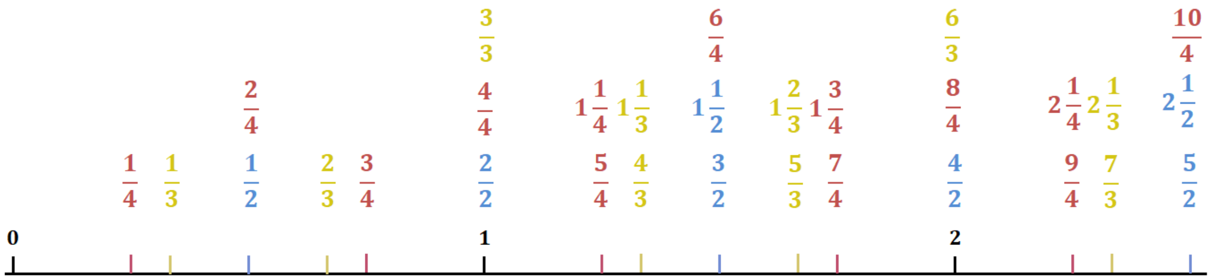

Figure 8: Fractional numbers populating the number line in different forms (Own elaboration)

We do not present here, due to space constraints, the development we have already carried out of the arithmetic operations involving these numbers, designed to make them clear and logical. Division allows us to verify that dividing two fractional numbers is always possible and exact, resulting in a fractional quotient. Furthermore, the operations provide fertile ground for expanding other roles that fractional numbers can play, such as acting as operators or ratios.

Observations on students' and teachers' reactions were made throughout the process, across different classes and topics. These observations helped guide our direction. Most of the time, students were engaged. When a few showed disinterest, we sometimes witnessed a firm response from a peer saying, *"Go ahead, it really works"*. Genuine judges evaluating our proposals.

Some teachers expressed uncertainty: they acknowledged that students struggled with the topic, but they themselves felt comfortable with the rules they had learned. And the groups were never brought together: the teacher training sessions and student classrooms were always separate. At most, a teacher would try something out with their own class.

We also made some necessary retreats. We reduced the use of the manipulatives created in the *New Mathematics curriculum from 1st to 8th grade* project. These were strips divided into halves, fourths, eighths, etc., used for activities and games. They were appealing as games, but the equivalences had little impact on actual mathematical learning. Another retreat involved the illustrations used in the project's workbook. These featured MDF furniture with fronts divided into equivalent or related parts. We observed that this was still a confined model, just a single example drawn from reality.

7 Final Considerations

This text was written with the hope of reaching teachers and future teachers, although it was presented to curriculum researchers. Many other texts have already been written by the first author on the teaching and learning of fractions, based on her research and directly addressed to teachers. However, we observe that: a) little or nothing has changed in teaching practices; and b) research, instead of being designed based on the findings of prior studies, often ignores some work, cherry-picking what to include in their literature reviews. As a result, there is neither coherent growth in the body of knowledge in Mathematics Education nor consistency in the teaching proposals that research might influence. The research methodologies used are also heavily dictated by publication policies and by the instrumental rationality adopted by funding agencies.

The tone of what we have written for teachers (Bertoni, 2020) may be somewhat naïve, but the ideas are not. The goal has always been to increasingly unveil the universe of fractions, examining how far we could go, equipped with the part-whole interpretation. The trajectory narrated demonstrates the robustness of the part-whole approach for teaching and learning fractions when properly developed. It allows fractions to be understood as words and numerical symbols that quantify collections of real-world objects — collections that involve whole items and equal parts of the same item, thus expressing quantities.

Therefore, they acquire the status of numbers. This approach also naturally leads to recognizing fractional numbers as the quotient of two natural numbers, where the divisor is not zero. Our trajectory touched on the notions of measurement and ratio — students, for example, inferred that having 1 out of 4 was better than having 1 out of 5. However, these two concepts, both connected to fractional numbers, still need to be revisited.

Students were also able to perceive that each fractional number has multiple representations, and that all of them represent the same number. As for the number line, they felt as though they had covered it completely. But that is only an impression.

Our search for literature on the teaching of fractions grounded in real-world contexts, without relying on geometric models, proved unfruitful. Nevertheless, we believed that the IOWO Institute in the Netherlands might offer relevant material. The second author then found the book *Fractions in Realistic Mathematics Education* (1991), by Streefland, a work from the Dutch group. Accessing the book was difficult and only became possible after the completion of the BNCC collection *Professores para Professores*, to which the first author contributed with the module on fractions (Bertoni, n.d.). That module already incorporated our main objectives: to work with fractions based on partitions of real-world objects and collections, to highlight their relationship with quantification (including in non-discrete sets), to develop toward the notion of number, and to construct the infinite set of these numbers on the number line, exploring their density.

In Streefland's book, we found an approach that aligned with and expanded upon our developmental research. The author describes the development and testing of a program for teaching fractions in primary school, already in practice at the time, and proposes an instructional theory within Realistic Mathematics Education. In Chapter 4, even without detailing how fractions were introduced to students, Streefland (1991) presents a wide variety of formal and informal activities, mobilizing different representations.

A striking example is the problem of three pizzas for four students. We had already discussed both the classical solution and the approach observed in our research, which allowed a historical connection with the Egyptian method. Streefland, in turn, expands the problem by considering different realistic ways of dividing the pizzas, simulating plausible cuts in a pizzeria setting. He mentions that three students could receive three-quarters of a pizza each, leaving three one-quarter slices for the fourth student — a distribution possibly unsatisfactory for that student. Another possibility is that two students receive three connected quarters, while the remaining two share what is left, each getting half a pizza and a quarter.

In another case, the author presents the division of five pizzas among eight students, where each receives $\frac{1}{4}+\frac{1}{4}+\frac{1}{8}$, and he asks how the pizzas were cut and served. The solution must be practical and realistic, avoiding excessive cuts or unusual shapes. This concern with the fair distribution of parts echoes the concerns of the ancient Egyptians (Gillings, 1962) and adds a meaningful layer to the problem.

The situations explored by Streefland, including tiling, mixtures in different proportions, distances, and debates in committees, are all grounded in real-world contexts and encourage students to think, share ideas, and argue. His approach represents a unique contribution to our proposal and to the continuation of this line of research.

We wondered whether the Realistic Mathematics Education (RME) curriculum was being implemented in the Netherlands. We were saddened to learn that the realistic Mathematics curriculum has been under attack by reactionary forces (Van den Heuvel-Panhuizen, 2010). These opponents are proposing yet another Mathematics Education reform — one that is nearly the polar opposite of RME. According to the critics, Mathematics should not be taught in context, informal strategies should be avoided because they confuse children, progressive schematization leads to a long and unnecessary detour, and the focus should not be on understanding — because, they argue, understanding will come automatically after drill. Moreover, they shamelessly claim that children do not need to think. In the minds of those attacking the RME reform, the main content to be learned in primary school should consist of written algorithms.

Marja Van den Heuvel-Panhuizen explains that the first forty years of implementation of the RME (Realistic Mathematics Education) curriculum occurred through trial and error, but in relative peace. In fact, it was a silent revolution, causing barely a whisper in the media. There was very little opposition and no pressure from above. She explained that during those forty years, from 1960 to 2010, the Ministry of Education was involved only in a facilitating role, without any significant interference regarding the content of the reform.

> Government subsidy made it possible that an extensive infrastructure arose in the Netherlands allowing development, research and training to take place in mutual coherence and cooperation with the field of education. Where other educational researchers were blamed for the gap between their research and educational practice, we were held up as an example of how research should be done; see the report by the Education Council of the Netherlands (Onderwijsraad, 2003). Not only was there recognition in our own country, but we also inspired developments in many other countries. Our work was, and still is, in great demand all over the world, even if only perhaps that it gives these countries good hope of being able to attain such high test scores as the Dutch. (Van den Heuvel-Panjuizen, 2010, p. 1)

Then, in 2004 and 2005, the first reports emerged showing lower results than in previous years, both in Dutch national tests and in international assessments such as International Results in Mathematics and Science (TIMSS) and Programme for International Student Assessment (PISA). And RME (Realistic Mathematics Education) became the target of criticism. Van den Heuvel-Panhuizen explains that the ensuing debate was far from academic — it was in fact a true smear campaign, carried out mainly through newspapers and websites. Many of the points of attack were not even truly characteristic of RME, as discussed in Van den Heuvel-Panhuizen (2010).

As the story unfolded, those who had made RME the scapegoat for the declining results in 2004 and 2005 decided to enter the market and publish a mechanistic textbook. They found a publisher initially interested in the project. However, the publisher sent the manuscript out for teacher review, and fortunately, as a result, the book's publication was rejected.

This episode leads us to reflect on how much standardized testing and the textbook market end up dictating what happens in schools. These forces often exert an influence and power that surpass the work of researchers and mathematics educators.

Conflicts of Interest

The authors declare no conflicts of interest that could influence the results of the study presented in the article.

Data Availability Statement

The data collected, produced, and analyzed in the article will be made available upon request to the authors.

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