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Educação Matemática Debate

Exponential Function: a proposal for mathematical tasks

Abstract: The purpose of this work has been to propose mathematical tasks involving the exponential function using the GeoGebra software. Thus, it has been based on the discussion about mathematical tasks by João Pedro da Ponte and the Theory of Semiotic Representation Registers. This is a qualitative explanatory research as regards its aims and it has been based upon bibliographical survey and assessement of the sources. Its development has involved the elaboration and adaptation of tasks through testing in the GeoGebra, followed by the writing of a set of didactical guidelines to teachers. The GeoGebra has been proved to be a suitable didactical resource for proposing tasks on exponential functions by enabling an experimental character in the classroom, which could provide High School students with autonomy in studying this mathematical concept.

Keywords: Exponential Function. GeoGebra. Mathematical Tasks. Semiotic Representations.

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Article

Función Exponencial: una propuesta de tareas matemáticas

Resumen: El objetivo de este trabajo fue proponer tareas matemáticas que involucraran la función exponencial utilizando el software GeoGebra. Para ello, se fundamentó en el debate sobre tareas matemáticas de João Pedro da Ponte y en la Teoría de los Registros de Representación Semiótica. Esta es una investigación cualitativa explicativa en cuanto a sus objetivos y bibliográfica en cuanto a sus procedimientos. La construcción de la propuesta implicó el desarrollo y la adaptación de tareas mediante pruebas en GeoGebra, seguido por la elaboración de orientaciones didácticas para el profesor. GeoGebra demostró ser un recurso didáctico adecuado para proponer tareas sobre funciones exponenciales al permitir un carácter experimental en la clase, lo que podría brindar autonomía a los estudiantes de Secundaria en el estudio de este concepto matemático.

Palabras clave: Función Exponencial. GeoGebra. Tareas Matemáticas. Representaciones Semióticas.

Função Exponencial: uma proposição de tarefas matemáticas

Resumo: O objetivo deste trabalho foi propor tarefas matemáticas envolvendo a função exponencial com o emprego do software GeoGebra. Para isso, fundamentou-se no debate sobre tarefas matemáticas de João Pedro da Ponte e na Teoria dos Registros de Representação Semiótica. Esta é uma pesquisa qualitativa explicativa quanto a seus objetivos e bibliográfica quanto aos seus procedimentos. A construção da proposta passou pela elaboração e adaptação das tarefas por meio de testagem no GeoGebra, seguida pela composição das orientações didáticas ao professor. O GeoGebra se mostrou um recurso didático adequado à proposição de tarefas sobre funções do tipo exponencial por possibilitar um caráter experimental à aula, o que poderá conceber ao aluno do Ensino Médio a autonomia no estudo deste conceito matemático.

Palavras-chave: Função Exponencial. GeoGebra. Tarefas Matemáticas. Representações Semióticas.

1 Introduction[1]

The concept of function is one of the most explored mathematical topics in Basic Education. Although the most systematic work with teaching its content occurs in High School, the BNCC (Brasil, 2017) reinforces that its intuitive notion may even be explored still in elementary school, through simple tasks involving the relationship between proportional variations of quantities (such as, for example, when comparing an amount of sugar in order to prepare a given quantity of juice).

Regarding the exponential function, some researchers and teachers report, however, difficulties faced by High School students in understanding the subject in regard to such aspects as: recognizing it in two distinct representations and even operationalizing it in its algebraic form. (Santos and Bianchini, 2012; Silva, 2016; Coelho, 2016; Silva and Lazzarin, 2018; Goldone, 2019; Silva, Prando and Gualandi, 2020). In addition to the exponential function, algebraically defined by $f(x) = a^x$ with $a>0$, there are the so-called exponential type functions written as $g(x) = b \cdot a^{c \cdot x} + d$ which will be discussed in this paper. Even though they do not comply with all the characteristics and properties of the exponential function, this type of function has its relevance underlined, mainly, by the manipulation of its coefficients, which enables a more global study of exponential behavior (Faria, Souza Junior and Cardoso, 2016). From the perspective of semiotic representations, the g graph has four visually distinct configurations, as illustrated in Table 1.

From a didactical point of view, Piano (2016) reinforces that the concept of exponential function is commonly undermined in the school syllabus, in which students present numerous learning difficulties. For the author, such difficulties often arise from insufficient time for a good approach/exploration of the concept in the classroom, which ends up being overlooked in relation to the other types of functions.

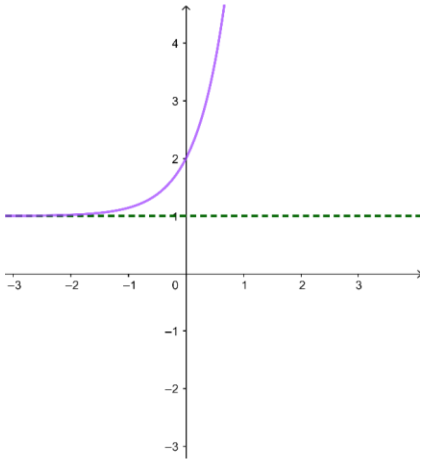
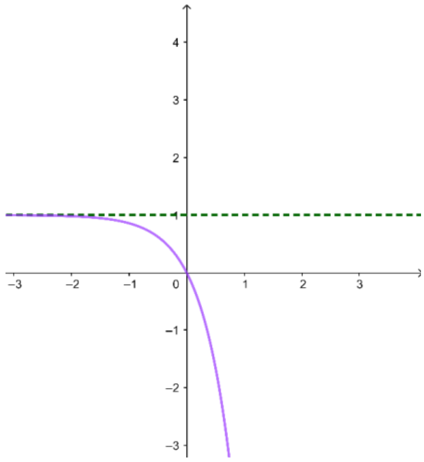
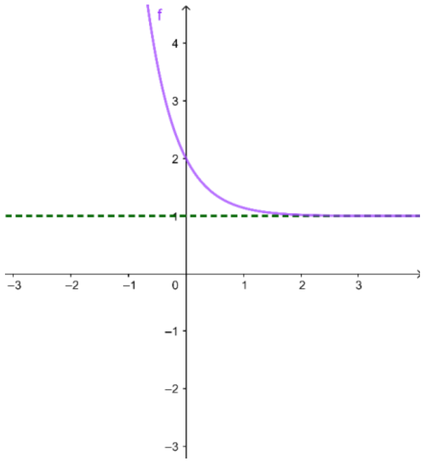
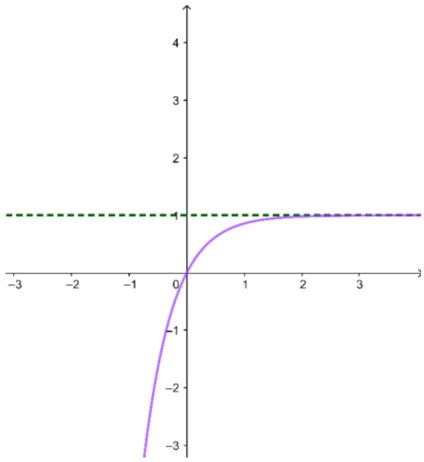
In any case, in the same group as the similar and the quadratic functions, the exponential function is one of the most discussed types in High School (Lima, 2013), and this recurrence may be justified by its numerous applications as a model for describing, predicting and analyzing a series of natural phenomena such as the amount of compound interests, temperature variation, learning curves, atmospheric pressure and the growth/reproduction of bacteria and viruses — such as the Covid-19 pandemic behavior as a function of time.

Nevertheless, corroborating Piano's (2016) findings, data from national evaluations reveal low student performance at progressive levels of proficiency[2] that require skills on exponential functions. The Basic Education Assessment System — SAEB (Brasil, 2019) reveals that only 1.11% of High School students in Brazil reached level 7, which includes the ability to solve problems involving exponential functions. At this same level, in Bahia, it reached 0.57% and in the city of Barreiras (BA) — where the authors of this article live, work and research — the number drops to 0.24%. Regarding level 9, which includes the ability to determine the algebraic expression of an exponential function from a text or graph, only 0.07% achieved it in Brazil, 0.03% in Bahia and 0% in Barreiras.

[1] This article is part of the master's thesis defended in the Programa de Pós-Graduação em Matemática (PROFMAT) at the Universidade Federal do Oeste da Bahia (UFOB), organized in multipaper format, written by the first author and supervised by the second author.

[2] "The results of the learning tests carried out are presented on a proficiency scale, compounded of progressive and cumulative levels, from the lowest to the highest proficiency level. This means that when a percentage of students are positioned at a certain level of the scale, it is assumed that, in addition to having developed the skills related to this level, they probably also had developed the skills related to previous levels" (Brasil, 2019).

Table 1: Visually distinct graphical representations of $f(x) = b \cdot a^{c \cdot x} + d$

Case 01: $b > 0$, $a > 1$ e $c > 0$ ou $b > 0$, $0 < a < 1$ e $c < 0$ 	Case 02: $b < 0$, $a > 1$ e $c > 0$ ou $b < 0$, $0 < a < 1$ e $c < 0$ 
Case 03: $b > 0$, $0 < a < 1$ e $c > 0$ ou $b > 0$, $a > 1$ e $c < 0$ 	Case 04: $b < 0$, $0 < a < 1$ e $c > 0$ ou $b < 0$, $a > 1$ e $c < 0$ 

Source: Own elaboration

Based on those data, we conjecture that one of the justifications for such low performance, and as already mentioned by Coelho (2016), is the time lapse that this topic is left out of the school syllabus or, even, the teacher's difficulty in teaching it. In this sense, Santos and Bianchini (2012) evoke the need for diversification of methodologies on the part of the teachers, claiming that the use of the textbook alone does not meet the students' needs regarding this matter while, according to them, the GeoGebra software is an important instrument of mediation.

In that context, Digital Information and Communication Technologies are, in turn, resources with wide discussion among researchers in the field of Education in Mathematics, which can contribute to overcoming those problems related to the teaching of exponential functions. Software from Mathematics and, in particular, from dynamic Mathematics, such as the GeoGebra, has the general characteristics of providing Mathematics classes with possibilities of simultaneous manipulation, interaction and analysis of a given mathematical object, especially the functions.

It is important to emphasize the need for teachers' didactical and technical training regarding the use of a certain technological resource, so that it is truly possible to explore the teaching possibilities of that methodology. Many teachers, although recognizing the importance of such tools and, in some cases, even stating that they use them in the classroom, yet, have a superficial and limited view of their didactical use, which does not allow them to carry out more efficient explorations in the classroom (Carvalho, 2017).

Using a multimedia projector, a slide presentation or the GeoGebra itself, sporadically, without planning and basic didactical mastery of both the program and the computer's operating system itself, are such instances that weaken the class and can place the teacher in an uncomfortable situation. Such a situation can be highlighted, for example, when a simple technical problem or a more incisive and complex question arises from the student about how to use a resource Y to analyze a problem X.

Thus, the GeoGebra can be an important teaching resource in the classroom, as it can give Mathematics classes an experimental character that allows students to compare, mainly, the algebraic and graphical representations of mathematical objects (Hohenwarter and Fuchs, 2004).

Some scholars in Brazil (Rezende, Pesco and Bartolosi, 2012; Santos and Bianchini, 2012; Silva, 2016; Faria, Souza Junior and Cardoso, 2016; Martins, Doering and Bartz, 2017; Sousa and Ramos, 2017; Silva and Lazzarin, 2018; Goldoni, 2019; Ferreira and Pereira da Costa, 2021) dedicated themselves to researching the didactical effects of methodologies that combined the teaching of the exponential function associated with the use of the GeoGebra.

Such studies make it clear that, in general, one of the main characteristics of the program is the empirical character that it can impart to the class, instilling in the students the autonomy and the possibility of working concomitantly with the multiple representations of the function.

Thus, the purpose has been to propose mathematical tasks involving the functions of exponential type using the GeoGebra software. In order to accomplish that, we have anchored ourselves in studies on the concept and classification of tasks by Ponte (2005) and on the Theory of Semiotic Representation Registers by Duval (2017). The aim has been to highlight the didactical aspects that teachers should take into account, such as planning, readings and technical/didactical skills, besides the way in which the software can be used to explore specific skills related to exponential functions.

2 Mathematical tasks

Ponte (2005) is a college professor and a researcher in Mathematics Education who dedicates part of his studies to discussing the concept of task, worrying about the forms of intervention and teaching objectives that generate, according to the author, different demands depending on the educational context in question. Forthwith it is stated that the discussion about tasks is relevant only if the teaching takes into account the active role of students, whereas the tasks are the organizing elements of their activity (Ponte *et al.,* 2015). A task is a mediation tool that may be either elaborated by the teacher and proposed to the student or may be put forward by the student and discussed with the teacher. It also may be proposed either at the beginning of the class or be formulated throughout it (Ponte, 2005).

In general, a task is an objective of an action or of an activity that, in turn, refers to what the student does in a given context, that is, a task can generate several activities. In order to accomplish that it is necessary to take into account: the way it has been proposed (at the beginning or at the end, ready-made or formulated during the class); the students' profile (whether they are more speculative, for example); the school environment and the teacher's own experience (Ponte, 2014). So, for example, *the question 27 of chapter 7 about exponential function* is a task and the way the student will solve it, the discussions he has with both the teacher and his classmates, the tools (the GeoGebra, for example) he has used for the resolution and the way he has justified and has interpreted his answer, all of which constitutes an activity.

Having defined the concept of a task, Ponte (2005, 2014) establishes its classification and typology which are based upon some aspects. The most fundamental among them are the degree of challenge and structure. The first refers to the perception of difficulty that the student will have when faced with a given task, which fluctuates within the high and low poles. The degree of structure may be either closed — when both, what is being asked as well as the information given in the question, are clear and objective and the student will have the mission to immediately apply a concept — or open — which involves some indeterminacy whether in the information within the statement or in the answer itself possibly compelling the student to deeper analyses, inferences and conjectures to find the best way of solving the task and its most appropriate answer. Based on these aspects, we have four types of tasks, as illustrated in Table 2.

Table 2: Types of tasks based on their degree of challenge and structure

	Degree of **High** Challenge	Degree of **Low** Challenge
Open Structure	Investigations	Explorations
Closed Structure	Problems	Exercices

Source: Adapted from Ponte (2005, p. 8)

Even though that demarcation, Ponte (2005) reinforces, with regard to the degree of challenge, that the main aspect distinguishing an investigation from an exploration and a problem from an exercise is the fact whether or not the student previously possesses the tools — technical and cognitive — for solving the task. If he doesn't, he will be facing a question with a high level of challenge, otherwise, the level will be reduced. Therefore, the importance of the epistemological baggage the student brings from home, as a result of the experiences, as well as the knowledge produced in other subjects or in other series and units, is reinforced, thus highlighting the hierarchical character and the transversality of Mathematics.

Another aspect that Ponte (2005, 2014) adds to classify the above tasks is the context which has as its extremes the poles of reality — use of exponential functions in Financial Mathematics, Biology and Chemistry, for example — and pure mathematics — operations and demonstrations with the use of strictly algebraic techniques and properties, for example. Skovsmose (2000, *apud* Ponte, 2005) considers yet an intermediate level given by semi-reality, which includes those tasks that are pseudo-real, as they are utopian — *a man takes a car journey that perfectly describes the following parable…* — or because discriminating excessively against some properties and external factors in favor of the interpretation and the resolution of the question — *... calculate the traveling time considering the entire path on a straight line, a constant speed and traveling without stopping…* —. Thus, we have Figure 1.

Figure 1: Classification of questions by their context (Ponte, Quaresma and Branco, 2011, p. 11).

Ponte (2005) also reinforces that it is not possible to associate a specific type of task with a given context, as it is possible to have problems which are centered on pure Mathematics and exercises in a real context, for example.

3 Theory of the registers of semiotic representation

This theory, created by Duval (2012a), is based on the premise that learning Mathematics necessarily involves the understanding and the operationalizing of representations. That is justified by the difference between Mathematics and other areas of knowledge whose objects being studied are non-semiotic and, therefore, accessible through implements (a telescope for astronomy, for example). Mathematical objectives are semiotic, and it is only possible to access them through their multiple representations (Moretti, 2002; Duval, 2017; Pereira da Costa and Rosa dos Santos, 2020).

In that sense, a mathematical object can be the exponential function, a circle or the number one, while its representations can be given by $f(x) = a^x$, *a set of points on the plane equidistant from a fixed point* and $\log_2 2$, respectively. Understanding that difference and knowing how to deal with it is a necessary and relevant task for the Mathematics teacher. Discussions about students who do not recognize a given mathematical object in more than one representation are common (Santos, 2012; Silva, 2016), such as, for example, not knowing how to identify the growth curve of COVID-19 cases as a representation of the concept of exponential function, recognizing it just by its algebraic law. Another situation is that, for the sake of optimization, it is easier to use the graph of a function to carry out comparative analyzes and projections than its algebraic form, for example.

In general, the fact of not confusing a mathematical object with its representation while identifying this same object in several different representations generates what Duval (2014) calls a *cognitive paradox*. Before presenting the solution given by his theory to overcome such a paradox, the researcher also adds a third factor in the relationship between a representative and a represented, which are the significant units. Such units are the characteristic elements of each representation that carry with them their own contents endowed with properties — such as variables, coefficients, traits, curvature etc.

It turns out that, in the classroom, more attention will be paid to mental representations (the concept that a student has about a subject) than to semiotic representations (Duval, 2012b; Hillesheim and Moretti, 2013). To designate different types of semiotic representations in Mathematics, Duval (2017), while paraphrasing Descartes, uses the term register, of which we highlight: the natural language, the algebraic and the numerical writing — binary, decimal and fractional, for example —, and the graphic representations. When working with functions, it is also common to use table representations, associating magnitudes. The author also points out that to be considered a semiotic system, the register must allow three essential cognitive activities: the formation of an identifiable representation, the processing and the conversion.

The formation of an identifiable representation refers to a semiotic system endowed with rules and properties that make its recognition and operationalization possible, such as grammatical rules for writing systems and geometric rules for polygons (Duval, 2012b). That cognitive activity allows for the procedures to be carried out.

The processing is an internal transformation of a register that is conditioned by its rules. Calculation and reconfiguration are examples of processing of algebraic and geometric register, respectively (Duval, 2012b). It turns out that, in many cases, it is not enough to stay in the same register to solve a problem, as in the case where it is necessary to construct a statistical graph from a series of data placed in tables. In that case we have a conversion.

Conversion is an external transformation in which one moves from one register to another, preserving all or part of the contents of the initial representation (Duval, 2012b). The author reinforces that the conversion is a cognitive activity different from processing, as many students may, for example, know how to find Cartesian points of an exponential function as well as, with their graph ready, know how to interpret whether it is increasing or decreasing, but are unable to associate the graph with the algebraic expression or to understand when one is more useful than the other and carry out that transition, without being asked for it.

Conversion has its value, from a teaching point of view, as it is an instrument that allows a student to choose the best system for solving a task while it is an excellent analysis tool for the teacher to check whether the students understand what, for example, an exponential function is or whether he only recognizes it when presented algebraically, with specific coefficients and using the symbols x and $f(x)$ to designate its domain and image. Perhaps because it has not been explored in demonstration, in operationalization and in testing activities — in which processing stands out —, the conversion has not attracted as much attention (Hillesheim and Moretti, 2013). From the point of view of the Mathematics teacher who is analyzing the resolution of a task developed by a certain student, what should be taken into account? What criteria should be considered to analyze that resolution and, before that, to prepare that task? Here arises the need to understand the phenomenon of semantic congruence related to the transition among representation registers.

Semantic congruence refers to the association of significant units of departure and arrival register relating to the same mathematical object, which may be trivial — we say that there is congruence — or may require adaptations and interpretations for that to be possible — there is no congruence. Duval (2012b) establishes three criteria to analyze the congruence between two register, as shown in Table 3.

Table 3: Semantic congruence criteria between two registers

Criteria	Characteristics
The possibility of a ***semantic* correspondence** of significant elements	To each simple significant unit of one of the representations, an elementary unit can be associated.
Terminal ***semantic* univocity**	Each elementary significant unit of the departure representation corresponds to a single elementary significant unit in the register of the arrival representation.
The organization of significant units	The respective organizations of the significant units of two compared representations lead to understanding the units in semantic correspondence, according to the same order in the two representations. This correspondence criterion, in the order of arrangement of the units that make up each of the two representations, is only relevant when they have the same dime

Source: Adapted from Duval (2012b, p. 283-284)

It is important, therefore, for the teacher to pay attention to the tasks developed with his students, as those that require congruent conversions — such as constructing the graph of a function, based on its algebraic expression — can give a false impression of learning, when, in fact, it is the opposite (not congruent), which reveals their difficulties in recognizing the same object in two different representations (Duval, 2005 *apud* Hillesheim and Moretti, 2013).

In short, the solution proposed by Duval (2012b) for the cognitive paradox is to condition mathematical learning to the coordination and recognition, on the part of the student, of at least two registers of the same mathematical object. There will be signs of learning, on the part of the student, when he learns how to move between such registers (conversion), recognizing the relationship between their meaningful units in situations when there is and, mainly, when there is not a congruence, knowing how to perform the internal operations and transformations of each semiotic system (processing).

4 Methodological aspects

This is a qualitative research, in terms of its approach, as it is not concerned with numerical representation, but rather with deepening the understanding of the objectives studied (Gerhardt and Silveira, 2009). As for its objectives, it is explanatory, as the main characteristic of such explanatory research is the identification of the factors that determine or contribute for the occurrence of certain phenomena (Gil, 2002).

We focus on proposing different mathematical tasks about the concept of exponential function to be worked on in High School — in real classroom situations. Thus, emphasis was placed on what the teacher must foresee to plan accordingly and what the student is expected to employ through the use of previous knowledge in terms of theories (semiotic representations and classification of tasks), tools (GeoGebra) and other technical and scientific data (tasks adapted from other materials and empirical information for constructing models).

We present examples of the four types of tasks — exercises, problems, explorations and investigations — discussed by Ponte (2005). In order to accomplish that we rely on the assumptions of the Theory of Semiotic Representation Register, with regard to the need of coordinating multiple register of the same object, with the congruence relationship and the behaviors of the exponential function discussed in the previous topic. Regarding the classification of tasks, we present the qualities and precautions that the teacher must take when applying them, as well as the convenient times to do so.

Our purpose is that the present proposal serves as a pathway for teachers to discuss the contents of exponential functions, providing them with a guide to facilitate and assist their work through GeoGebra. Aware of such problems, as those cited by Piano (2016), regarding insufficient classroom time for a well-developed and detailed work, we foresee a total workload of about 6 hours/class dedicated exclusively to the subject.

As regards to the proposal for tasks, this study is composed of the following stages: 1st stage: elaboration and adaptation (a priori) of tasks; 2nd stage: testing on GeoGebra; 3rd stage: resumption of tasks (modifications made after the first test on GeoGebra); 4th stage: (re)testing on GeoGebra; 5th stage: preparation of guidelines on the use of tasks.

Therefore, we present two teaching proposals, which the teacher may include in the work on the exponential function — in terms of the order in which they will be presented and of the objectives printed at each stage — and, obviously, valid and welcome implementations that add to, and are in in line with, the fundamentals which have been laid down in this paper.

5 Proposal for tasks

This proposal is based on the concepts of exploratory teaching and learning defended by Ponte (2005), in which the teacher does not seek to explain everything, but rather leaves an important part of the work of discovery and construction of knowledge for the students to carry out. This model is marked by an emphasis on exploration and investigation tasks, although exercises and problems also have their relevance.

This method is a counterpoint to what the author calls direct teaching marked by the centrality of the teacher. In this case, even requesting student participation or proposing some open-structure tasks, there is sporadic active participation from the class. Thus asking students objective questions so that the answers are a specific result of a calculation, alternatives such as *yes* or *no*, or even analyzes of previously submitted procedures, do not take away the teacher's centrality as such.

That said, one possibility of introducing a new content is, instead of presenting definitions and properties immediately, to propose exploratory tasks to students. Such circumstances will allow them to carry out conjectures, reflections and debates on the subject so that, even though they might not reach the formal definition in a strict way, they still can approach it in an active and not a passive way.

This methodology requires the use of a computer laboratory or any environment — virtual, for example — in which everyone involved has access to a computer or cell phone with an installed GeoGebra. Furthermore, it would be interesting, if it is possible, that students have already had some contact with the program in a previous class.

5.1 Proposal 1

This is a proposal that brings together two sets of independent but articulated tasks, the first set having a closed structure (exercises and problems) and the second with an open structure (explorations and/or investigations). Before starting the proposal (which takes on the nature of a workshop) students need to be allocated to a context that will guide the entire experience. They will be presented with a situation that involves an exponential behavior associated with the pandemic situation caused by the Covid-19.

Table 3: Task with a closed structure

Task: Giant Water Lily (Victoria Amazonica)
In April 2020, the initial period of the coronavirus pandemic, a Brazilian researcher proposed a riddle to popularize the explanation of how the virus spreads and multiplies in exponential growth. Maurício Féo (engineer, master in Instrumentation from the Brazilian Center for Physical Research and Ph.D student in Particle Physics in Geneva, Switzerland) recorded a video making an analogy with giant water lilies, a lake, and the quantity of this aquatic plant that is possible to be taken from the lake over a period of time. To accomplish that, he has considered that, every day, each one of the plants reproduces, generating another giant water lily. Source: Information taken and adapted from https://g1.globo.com/bemestar/coronavirus/noticia/2020/04/10/enigma-da-vitoria-regia-vira-exemplo-em-video-que-explica-o-que-e-o-crescimento-exponencial-da-pandemia.ghtml Based on this context, answer the questions below: 1. Consider that, such as in the enunciation above, each water lily reproduces itself by generating another over the course of a day and that on the first day studying such reproduction there are five water lilies in the lake. How many water lilies were there in the early days? 2. In the GeoGebra, insert the coordinates given by the points (day, number of water lilies) and analyze their behavior. What is characteristic about the way those points behave? 3. Is it possible to think of an algebraic expression ("formula") that allows us to find the number of water lilies in that lake at any given time, without doing much work ? Insert the algebraic expression found in the GeoGebra and check whether, in fact, it describes the behavior of the points (day, number of water lilies). 4. At any given time there were 10,240 water lilies in the lake. Can you tell which day that has probably occurred?

Source: Own elaboration

Table 4: Task with an open structure

Task: Giant Water Lily (Victoria Amazonica)
In April 2020, the initial period of the coronavirus pandemic, a Brazilian researcher proposed a riddle to popularize the explanation of how the virus spreads and multiplies in exponential growth. Maurício Féo (engineer, master in Instrumentation from the Brazilian Center for Physical Research and Ph.D student in

Particle Physics in Geneva, Switzerland) recorded a video making an analogy with giant water lilies, a lake, and the quantity of this aquatic plant that is possible to be taken from the lake over a period of time. To accomplish that, he has considered that, every day, each one of the plants reproduces, generating another giant water lily.

Source: Information taken and adapted from https://g1.globo.com/bemestar/coronavirus/noticia/2020/04/10/enigma-da-vitoria-regia-vira-exemplo-em-video-que-explica-o-que-e-o-crescimento-exponencial-da-pandemia.ghtml

Based on this context, answer the questions below:

1. Consider the situation involving the reproduction of the water lilies of the statement and, hence, the algebraic expression that determines this reproduction is given by $f(x) = 5 \cdot 2^{x-1}$, with x being the days and $f(x)$ the corresponding number of water lilies. What happens to the algebraic expression and its graph if we vary the number of water lilies on the first day or the number of water lilies that one generates throughout each day?

Source: Own elaboration

The first question with a closed structure task has a reduced degree of challenge; therefore, it is an exercise. It is important to remember that the diversification of types of tasks is one of the points most discussed by Ponte (2005, 2011). Furthermore, the fact that a proposal is exploratory does not exclude the need for exercises and problems, which are important for the students' development of mathematical reasoning which, in turn, is based on a close and rigorous relationship between data and results, such as it is the case with that first question.

From the point of view of semiotic representations, it is a conversion of the representation in the natural language (in which the enunciation is found) to the decimal representation (in which the students must carry out their calculations, that is, they will carry out the treatments).

Students are expected to use intuitive techniques to find the correct answers to what has been asked. Students can think, as the enunciation indicates: on the first day there were five water lilies, on the second, as each one generated another, there will be ten... Another possible reasoning that may be used by the students is:

$$\textit{2nd day: } 5 + 5 \textit{ (generated from the others)} = 5 \times 2 = 10$$

$$\textit{3nd day: } 10 + 10 = 5 \times 2 \times 2 = 20$$

$$\textit{4nd day: } 20 + 20 = 5 \times 2 \times 2 \times 2 = 40$$

$$\textit{5nd day: } 40 + 40 = 5 \times 2 \times 2 \times 2 \times 2 = 80$$

$$\textit{7nd day: } 160 + 160 = 5 \times 2 \times 2 \times 2 \times 2 \times 2 = 320$$

That conversion (from the natural language representation to the decimal one), although it may be intuitive for some students, it is not congruent as it does not respect the semantic univocity. That is justified because the significant units of the initial representation *each one reproduces itself generating another* were transformed into the significant units *× 2* in the final representation (that is, in the representation of the ending point). Whereas the expression were *if doubled*, we could consider as the semantic congruence.

It is important to note that it is possible that students do not represent the solution as above, in despite of finding the same result. Thus, such arrangement in that way helps the generalization that will be required in a later task. Furthermore, after or during the conversion, students may make mistakes in algebraic treatments related to addition and multiplication operations. It is important, then, for the teacher to be attentive to this phase.

The second question with a closed structure, also in the form of an exercise, requires students to enter the points *day, number of water lilies*, found previously, into the GeoGebra. This is a congruent conversion (from the representation in ordered pairs to the graphical representation), as students will correspond the ordered pairs with the points on the Cartesian plane (construction of the point-to-point graph). It is expected that they will arrive at the following representation as shown in Figure 2.

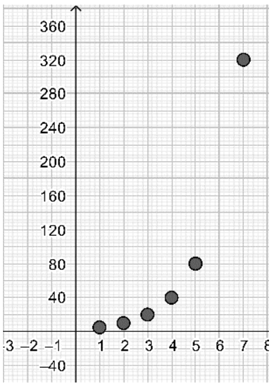


Figure 1: Projection of the points that associate the number of water lilies depending on the days (Own elaboration)

Still on this question, students will need to reflect, conjecture, compare and remember some of the properties relating to different types of functions, such as related and quadratic functions. It is expected that, immediately, students will notice the curved aspect of the behavior of the points and, based on that, rule out the possibility of it as a related function.

The GeoGebra has a feature called *slider controls* through which it is possible to insert a function with undefined coefficients. Thus, different functions can be inserted and controls can be created allowing for variant values of their coefficients and observing the effect of that simultaneously on the graph.

Here, students are expected to convince themselves that the growth behavior of water lilies can neither be linear nor quadratic — although the points describe a curved line —, an excellent cue for what is required next.

The third question of the closed structure reveals itself as a problem, both by its very evident demands on the students as its high degree of challenge. The students are expected to arrive at the model $f(x) = 5 \times 2^{x-1}$ (or $c(d) = 5 \times 2^{d-1}$) or, at least, its idea, so that even not finding the formal algebraic relationship, they can apply it at least intuitively to find the number of water lilies in any period.

Here we have a non-congruent conversion, because, although some significant units of the representation in natural language appear in a unique way in the algebraic representation — such as the initial quantity of five water lilies —, we also have the conversion of the expression in natural language *each one is reproduced generating another* for *× 2* in algebraic form and the *x – 1* in the exponent of the function which, in natural language, would be represented by *"previous day",* which is also not apparent in the enunciation.

The semantic non-congruity of that and other conversions required in this proposal, in despite of, in principle, denoting an obstacle, indicates an epistemological relevance to the task, as it gives to the experience a challenging character that gives rise to epistemological barriers. Those can be broken down by simultaneous mobilization of several registers of the same object, allowing students to recognize the same mathematical object in different semiotic representations (Duval, 2011, 2012b). Consequently, it will indicate the aspects that should be discussed and analyzed by the teacher as a way of understanding that relationship between departure and arrival records.

Trying to work on the subject only with treatments — *given the function, find f*(5) — or conversions congruent to a procedure for finding ordered pairs and representing them in the Cartesian plane — construct the graph of the function below —, it can give the students cause for a false notion of learning or make the entire activity, linked to the subject covered, banal — very similar to direct teaching, based only on exercises and problems.

In this same question, students must insert the functions discovered in the GeoGebra as a way of attesting and analyzing whether, in fact, such models perfectly describe the behavior of the points along the Cartesian plane. They are expected to exercise their autonomy and sense of self-evaluation to investigate their productions using the software to reflect and evaluate possible mistakes, in such a way as perceiving errors as teaching tools, and not as penalties.

Question four is a problem and requires an algebraic treatment of the model found in the previous task. Although it is a simple treatment, difficulties at this stage may arise. Students might be able to present the following development:

$$10240 = 5 \cdot 2^{d-1}$$

$$\Rightarrow \frac{10240}{5} = 2^{d-1}$$

$$\Rightarrow 2048 = 2^{d-1}$$

$$\Rightarrow 2^{11} = 2^{d-1}$$

$$\Rightarrow 11 = d - 1$$

$$\Rightarrow d = 12$$

As for the exploratory task, it aims to work with the fundamental issue relating to the understanding from the perspective of semiotic representations: recognizing the same mathematical object in two or more distinct representations, by discriminating the significant units of each one and by understanding that whichever the change or variation in one affects the other.

Here we present a possible solution from the perspective of emulating the way a student might think about solving this task. One of the main characteristics of the open structure tasks is precisely the opportunity for various ways of interpreting the resolution of the same task.

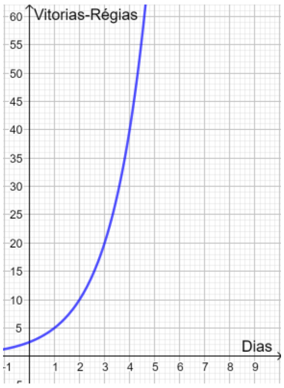
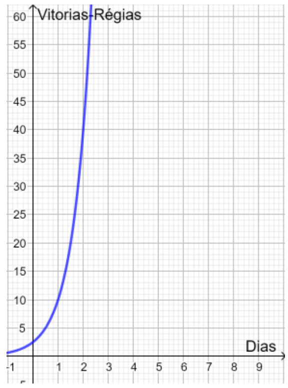
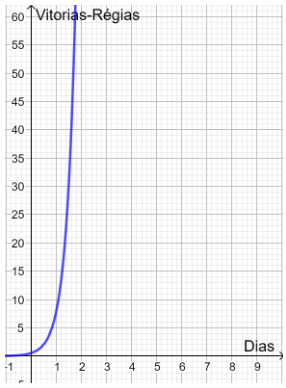
As stated a priori, this is an independent task that could even be applied without the previous ones if considering a class in which modeling was not the focus, but rather the study of the finished model. It is common in textbooks to have tasks that present ready-made functions based on which the students must perform what is being asked — to find an image, to build a graph etc.).

Considering, then, the function $f(x) = 5 \cdot 2^{x-1}$ with f being the number of water lilies as a function of days (x), it is expected that students understand the significant unit *5* as referring to the number of water lilies on the first day. Furthermore, we hope that they will notice that, if that number changes to six or seven, for example, they would simply replace such value in the algebraic expression.

Similarly, the significant unit *2* in the algebraic expression refers to the duplication implicit in the natural language of the enunciation and that if, instead of each one generating another (duplication), each one generated two others, we would have a triplication — and the number 2 would be replaced by the number 3 in the algebraic expression. The same reasoning would apply in the case of quadruplication, quintuplication etc.

Regarding the graphical expression of that function, students are expected to understand that changing the form of reproduction (triplication, quadruplication ...) or changing the number of water lilies (vitórias-régias) on the first day (dias) will modify the slope and monotonicity of the graph, as shown in some examples in Table 6.

Table 6: Graphical analysis of the variation of the significant units of the exponential function without its representation in natural language

<table>
<tr><td>5 water lilies on the first day with each one generating two others daily</td><td>10 water lilies on the first day with each one generating four others daily</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">8 water lilies on the first day with each one generating 15 others daily</td></tr>
<tr><td colspan="2">
</td></tr>
</table>

Source: Own elaboration

Understanding the relationship among those three representations is a way of inducing students to recognize the mathematical object *exponential type function* in several distinct semiotic representations, in an intuitive and non-mechanical way, so that whether changing the nomenclature of the variables — either $f(x)$, $c(d)$ or $g(n)$ —, or any coefficient or the order in which they appear, will not raise doubts or obscure their perception of the subject. That may be due to the fact that they have not studied it, being stuck with specific examples or in a class that favors extreme theory and determinism, to the detriment of practice and reflection.

5.2 Proposal 2

This is a proposal that involves two independent, but articulated, sets of tasks. The first, an investigative/exploratory task, and the second with closed structure tasks. Here students must study the process of heating a liquid to the room temperature, by means of Newton's Law of

Cooling.

Table 5: Open Structure Task

Task: Experiment with Newton's Law of Cooling
Newton's law of cooling determines that heat loss from a body is proportional to the difference between the body's temperature and the ambient temperature. Using this principle, each group will study a previously refrigerated glass of water placed at room temperature. There will be needed 150 ml of water (at a temperature of around 2° C) in a glass cup, a culinary thermometer and access to the GeoGebra. The environment must have a stable ambient temperature and must therefore be closed. 1. Insert the function $f(x) = b \cdot a^{c \cdot x} + d$ and using the GeoGebra, consider it and make notes in your notebook about the following aspects: a) What happens to the graph when we manipulate just the coefficient a? b) What happens to the graph when we manipulate just the coefficient d? c) What happens to the graph when we manipulate just the coefficient b? d) What happens to the graph when we manipulate just the coefficient c? 2. What else did you notice when carrying out this task?

Source: Own elaboration

Table 8: Closed Structure Task

Task: Experiment with Newton's Law of Cooling
Newton's law of cooling determines that heat loss from a body is proportional to the difference between the body's temperature and the ambient temperature. Using this principle, each group will study a previously refrigerated glass of water placed at room temperature. There will be needed 150 ml of water (at a temperature of around 2° C) in a glass cup, a culinary thermometer and access to the GeoGebra. The environment must have a stable ambient temperature and must therefore be closed. Using a thermometer, measure and record the room temperature. Then, measure and record the temperature of the liquid at fixed intervals (every 5 minutes, every 2 minutes, for example) for a period of 30 to 50 minutes. 1. Using a thermometer, measure and record the room temperature. Then, measure and record the temperature of the liquid at fixed intervals (every 5 minutes, every 2 minutes, for example) for a period of 30 to 50 minutes[3]. 2. Insert the points (momentum, temperature) and the function $f(x) = b \cdot a^{c \cdot x} + d$ into the GeoGebra and, using the coefficient adjustments in the GeoGebra, find a function that best models the behavior of the collected data. 3. Using the algebraic model proposed by Newton, find the function that models the behavior of the collected data.

Source: Own elaboration

The class should be divided into groups of three students while the teacher, after the students have already had contact with the definition and basic precepts of exponential functions — through proposal 1, for example —, should raise a discussion about Newton's cooling law (without displaying the model itself) and then should present to the students the exponential type function given by $f(x) = b \cdot a^{c \cdot x} + d$.

As it has been stated, functions with exponential behavior are models of a series of natural phenomena, such as the growth and reproduction of plants or bacteria, as we explored in the previous proposal. One of those applications is Newton's law of cooling, which considers the rate of change in temperature of a cooling body as a function of time $T(t)$, which is

[3] The breaks as well as the total duration of the experience (considering a minimum time of 30 minutes) are at the discretion of the teacher and the time availability. It is important, however, that everyone performs the measurements for the same period of time.

proportional to the difference between the body's temperature (T) and the constant temperature of the environment (T_m) (Zill and Cullen, 2001). In this way, we have the following relationship: $\frac{dT}{dt} = k(T - T_m)$.

Assuming the hypothesis that the temperature of the object depends on time and it is the same at all points of the observed liquid, that the ambient temperature is constant during the experiment and that the rate of temperature variation obeys Newton's cooling law (as above), by means of an Ordinary Differential Equation (ODE), we arrive at the following model $T(t) = C \cdot e^{k \cdot t} + T_m$, where C is the difference between the ambient temperature (T_m) and the initial temperature of the body and k is a constant of proportionality, which depends on the exposed surface, the specific heat of the body and the characteristics of the environment (Sias and Teixeira, 2006). Among the many applications of such a model there are the possibilities of estimating the time of a person's death and of predicting the moment when the milk will reach the ideal temperature (before it boils) when preparing a homemade yogurt.

This proposal has the character of exploratory teaching and learning proposed by Ponte (2005), in which the data will not be delivered ready-made but rather collected and each group will have the opportunity to create its own model in addition to being able to verify, with the teacher's help, whether in fact its function describes the analyzed behavior. Furthermore, this is also a proposal in the context of reality.

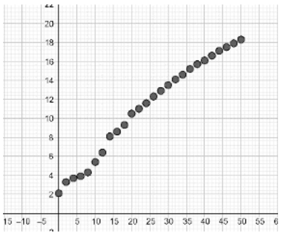
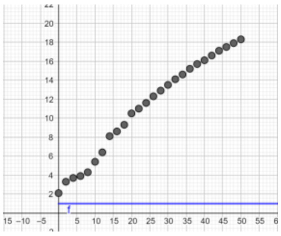
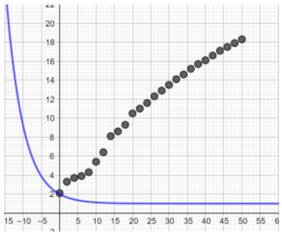
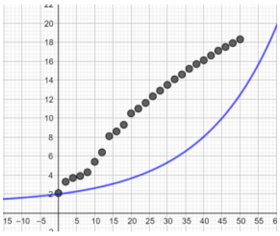
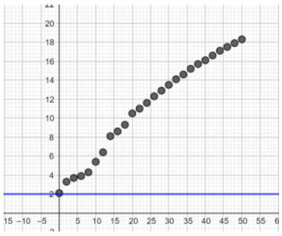
From the point of view of semiotic representations (Duval, 2011, 2012b), students will have the opportunity to work simultaneously with four semiotic representations, such as the tabular, the graphical, the algebraic and the natural language (when interpreting their results) representation. Next, we are going to describe this method which is based on the results of our own measurements with the same materials and conditions mentioned above.

As it is stated above, this is an independent task that might be applied in isolation without an application context, or it might be associated with other applications in other fields — such as Biology, for example. In the present investigative task, students will be faced with an open situation, about which they will have to make analyzes and conjectures from which they must find regularities, thus deducing important properties about the exponential function. The open structure is justified by the indeterminate nature of the answers and the way in which students may present them. It is expected that when analyzing the coefficients, the students take notes and try to explain, each in their own way, the behavior they see and the possible justifications for that. When controlling the coefficient *a,* while keeping the others equal to 1, for example, the students will be able to notice five distinctive behaviors, as seen in Table 9.

That is an opportunity of discovering and of discussing some of the properties applied to the base of that function, such as its condition of existence correlated to the impossibility of the negative base and the study of its monotonicity.

With the exponential format taken by the function, through the manipulation of a, a good opportunity arises to discuss the coefficient *d*. They should understand that, in its geometric representation, the variation of that coefficient causes a vertical translation and it represents a horizontal asymptote. From the point of view of representation in natural language — in relation to the temperature, for example —, students are expected to assimilate that d needs to be the direction of the temperature of the liquid over time, so that it remains nearby and stabilized, that is, in the ambient temperature, as it is seen in Figure 3, with the dotted green line being the horizontal asymptote.

Table 9: Correspondence between the graphical and algebraic representations based on the manipulation of the coefficient a

$a < 0$	$a = 0$	$0 < a < 1$
		
$a > 1$	$a = 1$	
		

Source: Own elaboration

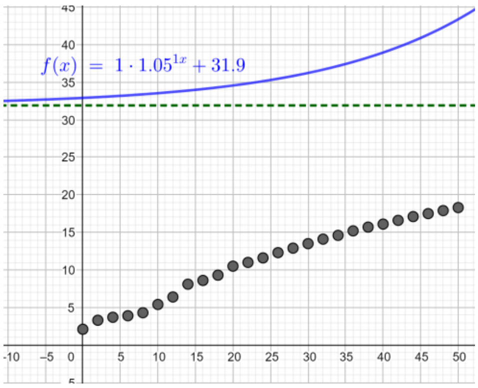


Figure 2: Graph of $f(x)$ to $a > 1$, $c > 0$, $b > 0$ e $d = 31{,}9$ (Own elaboration)

Coefficients b and c cause the same behavior in the graph: they change its slope and growth status, as does the coefficient a. From the perspective of an application, they modify the speed and way in which the studied phenomenon changes up to (or from) its stability (asymptote). Keeping the other coefficients fixed — $a = 1{,}05$, $c = 1$ and $d = 31{,}9$ — we can verify the influence of b on the graph, as in Figure 4.

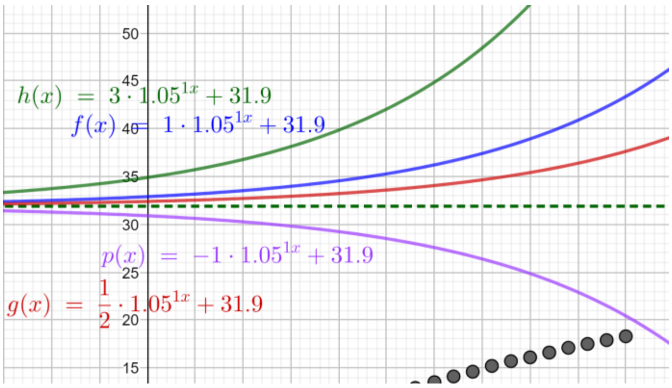


Figure 3: Different behaviors caused by the manipulation of b (Own elaboration)

Students will also be able to notice, analyzing $p(x)$ and $f(x)$, that the function undergoes a reflection around the asymptote for negative values of b. Furthermore, it is important to understand that for $b = 0$ we have $f(x) = d$ which is, therefore, a straight line. In general, the aim is for them to perceive three distinctive behaviors for b when it is negative, null and positive.

We may rewrite the function $f(x) = b \cdot a^{c \cdot x} + d$ as $f(x) = b \cdot (a^c)^x + d$, by using some potentiation properties. In this way, we have that c potentially modifies the base a and, therefore, affects the slope of the curve and its condition of growth, decreasing in a "faster" way. Keeping the other coefficients fixed — $a = 1{,}05$, $b = 1$ and $d = 31{,}9$ —, we can verify the influence of c on the graph in Figure 5.

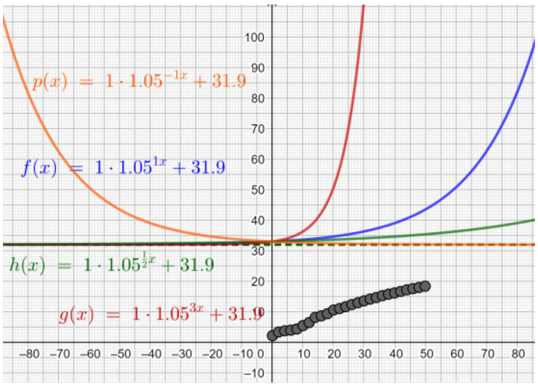


Figure 4: Different behaviors provoked by the manipulation of *c* (Own elaboration)

Students will also be able to notice that the negative values of c, while the other coefficients are constant, cause a reflection of the function around the ordinate axis (IF-USP, 2000), as evidenced by the comparison between the functions $p(x)$ and $f(x)$ in the previous image. In short, they are expected to write down three distinctive behaviors for c, whenever it is negative, null and positive.

A question is also asked to the students what else they observed when carrying out that task. It is an opportunity to encourage them to study the regularities and to work on their critical sense about whatever they have just done. Everyone will be free to express their views on how the task has been carried out while the teacher will be able to use such views to check whether the objectives of that first stage of the proposal have been met. Next, the closed structure task will be discussed.

We have used the Clink culinary thermometer (Figure 6), which has simple operation and controls, in addition to being of a low cost. In our experiment, the ambient temperature was 31.9° C.

Figure 5: Outset of the experiment (Own elaboration)

One possibility for organizing the data is, after its collection, to insert it into a table. Carrying out a simulation and recording the numerical data at two-minute time intervals each, as students would be able to do, we arrived at the following Table 10.

Table 10: Recording the water heating process as a function of time

Momentum (Min)	Temperature (°C)	Momentu (Min)	Temperature (°C)
0	2,1	26	12,3
2	3,3	28	12,9
4	3,7	30	13,5
6	3,9	32	14,1
8	4,3	34	14,6
10	5,4	36	15,2
12	6,4	38	15,7
14	8,1	40	16,1
16	8,6	42	16,6
18	9,3	44	17,1
20	10,5	46	17,5
22	11	48	17,9
24	11,6	50	18,3

Source: Own elaboration

Here, we see the treatment in the tabular representation, whose relevance lies in the organization and systematization of the data, so that it is possible to proceed to the next steps.

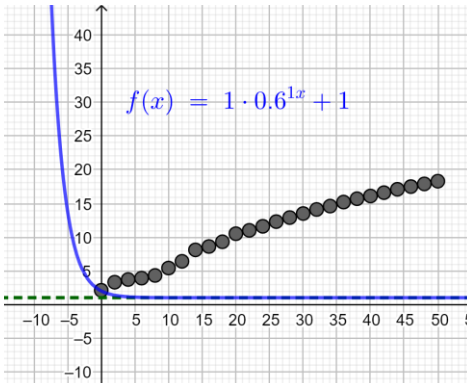
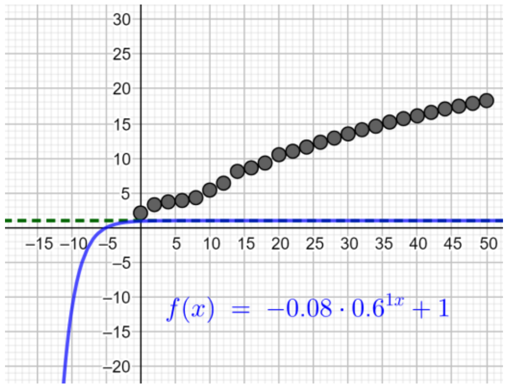
We have used the Clink culinary thermometer (Figure 6), which has simple operation and controls, in addition to being of a low cost. In our experiment, the ambient temperature was 31.9 °C.

By inserting the points (momentum, temperature) into the GeoGebra, students will be performing a congruent conversion from the tabular representation to the graphical one by using the point-to-point graph construction technique. In the GeoGebra, when entering the function $f(x) = b \cdot a^{c \cdot x} + d$, slider controls will be generated for the coefficients *b, a, c* and d. All of them will, initially, be equal to 1 by default. Such coefficients are significant units of the algebraic representation, whose manipulation allows comparative variations, related to the significance of the semiotic representations. That allows the analysis of the graph's behavior, when manipulating each coefficient, while keeping the others constant (Duval, 2012b). Thus, students will be able to establish the relationship between the graph and the algebraic expression, without being tied to mechanical processes, in addition to allowing them to analyze the different visual configurations that the exponential function might assume.

Finding the model by using the slider adjustments may be considered a problem, but the degree of challenge may be less after the students having studied the coefficients in the open structure task at the beginning of the proposal. At this point, students will have the freedom to manipulate the slider controls of the function's coefficients based on the properties which have been verified and discussed so far. It would be interesting if, at that stage, everyone come back to coefficients equals to 1. If any group should choose, in that case, to start by establishing 0 <

$a < 1$, they may follow the order of reasoning systematized in Table 11 in order to arrive at the desired visual configuration.

Table 6: A way to model a function to match the collected data by using the properties of the coefficients

Establishing $0 < a < 1$, we have: A decreasing function up to the asymptote	Then, putting $b < 0$, we have: A reflection around the asymptote and, therefore, an increasing function up to the asymptote
	

Source: Own elaboration

In that case, we already have the necessary behavior: the temperature increases until it stabilizes. What needs to be done now is an adjustment of the values of the coefficients, remaining in the Case 4 of the Table 1 (presented in the introduction of this article), that is, $0 < a < 1$, $b < 0$ e $c > 0$. They must also take into account the suitability of the coefficient d, such as the ambient temperature or the moment of stabilization. In this way, we can arrive at the following graph in Figure 7.

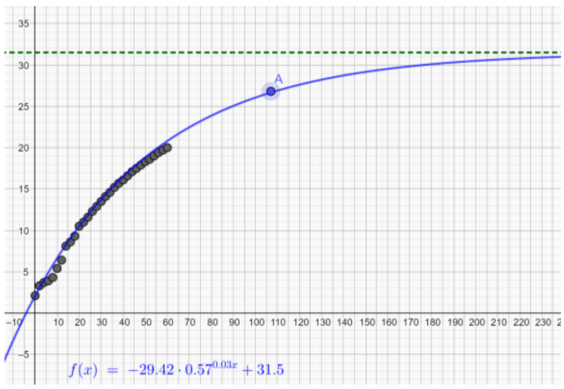


Figure 6: A proposal for an approximate model for the heating of the analyzed liquid (Own elaboration)

We have adjusted the coefficients in order to simulate what a student could do given that, until then, they would not know Newton's cooling law and the exact meanings of some of the coefficients — such as the base equal to e. Furthermore, in our experiment, we have monitored the liquid beyond 60 minutes, writing down, after the created function, the temperature state at the minute 107 — to verify the efficiency of our model. Thus, 26,8 °C has been recorded, which corresponds to the point A in the previous graph, reinforcing the degree of correlation between the function and the natural thermal behavior of water.

It is interesting, therefore, that students carry out that exercise to check the validity of their models both in relation to the data already inserted in the graph, as well as with the subsequent state of the liquid in question, given that the projection and prediction of data constitute one of the main objectives of modeling (Bassanezi, 2018).

The algebraic and graphical representations are closely associated with the study of functions, and evoke the analysis of semantic congruence between those two registers, a phenomenon much discussed by Duval (2011, 2012b). As it has been much debated, the transition from the algebraic to the graphical representation of the exponential type function can be done by means of what the researcher calls a point-to-point approach, configuring its semantic congruence.

In the inverse analysis, however, we realize, for example, that the monotonicity of the exponential curve is directly related to the behavior of three significant units of the algebraic expression (a, b and c). That indicates that the criterion of semantic univocity is not preserved in the analysis Graphic → Algebraic law, thus configuring the semantic non-congruence of that passage. In general, the conversion from graphical to algebraic representation requires a global interpretation (Duval, 2011).

Such an interpretation depends on the recognition of all the values of the graphic's visual variables and their corresponding symbolic expression. In this proposal, the simultaneous analysis of the correlation between the coefficient and the curve by using the GeoGebra allows the students to study such a relationship, exploring it on their own.

Next, we approach the formal work with the Law of Cooling itself, in which the students will become familiar with the algebraic expression and its real coefficients given by $T(t) = (T_\circ - T_m)e^{k \cdot t} + T_m$ where $T_\circ$ is the initial temperature of the body (in our case, 2.1°C) and T_a is the ambient temperature (31.9°C) with $k < 0$. The question assumes the status of a problem due to its closed nature and higher degree of challenge, whose main virtue is to offer students an effective mathematical experience.

In addition to the relevance and need for time spent in exploratory practices, Ponte (2005) also highlights the importance of reflection and debate about the mathematical object with which one is working. That is necessary so that there is no risk of important information not being highlighted or that students might become confused about what, and for what purpose, they are learning.

When presented with Newton's model above, a dilemma may arise: that function (thinking about the model $f(x) = b \cdot a^{c \cdot x} + d$) is defined for $a = e$, that is, $a > 1$, in addition to $c < 0$, whereas in our model found in figure 7, we have $0 < a < 1$ e $c > 0$. Such as we have done, some group may arrive at that same result, although it is also possible that, experimentally, they arrive, approximately, at Newton's model. The answer to that is in table 1, shown in the introduction of this article, in which we present the visually distinct graphical configurations of the exponential type function. In case 4, we state that such behavior (growth up to an asymptote) occurs in two cases. One of them was found in our manipulation ($0 < a < 1$, $b < 0$ and $c > 0$) and the other is the one which corresponds to Newton's law ($b < 0$, $a > 1$ e $c < 0$).

In addition to the fact that such equivalence may be verified graphically by using the slider controls, it is also possible to verify it by means of an algebraic treatment of the function f that has been found (Figure 7). Therefore, let us apply the base change to f, disregarding the constant $d = 31{,}5$, as it is the same in both cases (since, in any case, the ambient temperature cannot change). Thus, we have $h(x) = y = -29{,}42 \cdot 0{,}57^{0{,}03 \cdot x}$. By applying ln on both sides of the equation, we obtain:

$$\ln y = \ln(-29{,}42 \cdot 0{,}57^{0{,}03 \cdot x})$$

$$\Rightarrow \ln y = \ln -29{,}42 + \ln 0{,}57^{0{,}03 \cdot x}$$

$$\Rightarrow \ln y = \ln -29{,}42 + 0{,}03 \cdot \mathrm{x} \cdot \ln 0{,}57$$

$$\Rightarrow y = e^{\ln -29{,}42 + 0{,}03 \cdot x \cdot \ln 0{,}57}$$

$$\Rightarrow y = e^{\ln -29{,}42} \quad \times e^{0{,}03 \cdot x \cdot \ln 0{,}57}$$

Using the property that states that $e^{\ln a} = a$ and that $\ln 0{,}57 \cong -0{,}5621$, we find

$$y = -29{,}42 \cdot e^{-0{,}01686 \cdot x}$$

So, $g(x) = -29{,}42 \cdot e^{-0{,}01686x} + 31{,}5$ is a function algebraically equivalent to $f(x)$, as shown in Figure 7.

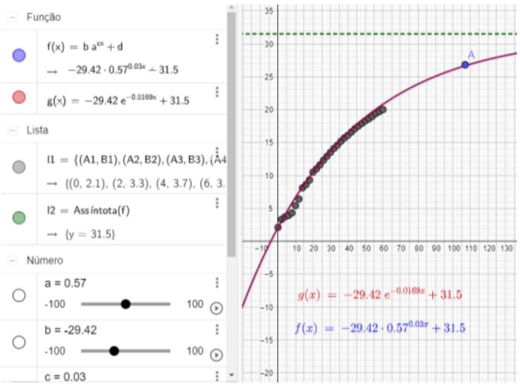


Figure 7: Graphical comparison of functions $f(x)$ e $g(x)$ (Own elaboration)

Finally, students can find such a model by using strictly the algebraic cooling function. Here, they will even be able to see that the coefficient b represents the difference between the ambient temperature and the initial temperature of the cooled body, which is another correspondence between the algebraic and natural language representations (enunciation) of that object. Substituting the other coefficients and using the point $(30, 13.5)$ to find the constant k of the law, we have:

$$(2{,}1 - 31{,}9) \cdot e^{30 \cdot k} + 31{,}9 = 13{,}5$$

$$-29{,}8 \cdot e^{30 \cdot k} = 13{,}5 - 31{,}9$$

$$\Rightarrow e^{30 \cdot x} = \frac{-18{,}4}{-29{,}8}$$

$$\Rightarrow \ln e^{30 \cdot k} = \ln \frac{18{,}4}{29{,}8}$$

$$\Rightarrow 30 \cdot k \cdot \ln e = \ln \frac{18{,}4}{29{,}8}$$

$$\Rightarrow 30 \cdot k = \ln \frac{18{,}4}{29{,}8}$$

$$\Rightarrow k = \frac{\ln \frac{18{,}4}{29{,}8}}{30}$$

$$\Rightarrow k \cong -0{,}01607$$

Therefore, the cooling function, using the formal law, will be given by

$$p(x) = -29{,}8 \cdot e^{-0{,}01607 \cdot x} + 31{,}9$$

graphically represented, such asFigure 8:

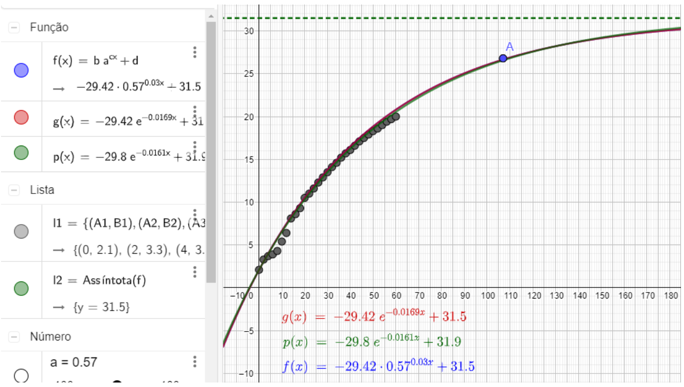


Figure 8: Graphical comparison among the three functions found for the analyzed thermal behavior (Own elaboration)

Those last algebraic treatments require, in fact, the application of some of the properties of powers and logarithms. Therefore, its application depends on a series of factors related to the class and their prior knowledge. This proposal is recommended, however, to close the subject, after, for example, the application of proposal 1 presented here, when the students have already had some basic previous contact with the functions, with exponential behavior. It may also be applied before or during the study of the logarithmic function, given that, generally, it is subsequent to the exponential function and it maintains a close relationship with that as it is its inverse.

6 Final considerations

The objective has been to propose mathematical tasks involving exponential type functions with the use of the GeoGebra software. In order to accomplish that, we have been based on the studies on the concept and classification of tasks by Ponte (2005) and on the Theory of Semiotic Representation Registers by Duval (2017).

From the point of view of the proposition of the tasks, we believe that one of the main aspects that teachers must take into account when planning and executing any intervention in the classroom — from the most common class to those involving experiments — is the nature and diversification of the tasks they will apply. In doing so, it is necessary to consider the proposed topic, the learning objectives, the competence and skills they want to explore, in addition to considering the cognitive and social characteristics of the students with whom they will work.

We focus on the types of tasks defined by Ponte (2005, 2014), which considers the existence of specific qualities related to the use of each task, such as those of a more accessible nature (explorations and exercises). Such fact allows students a certain degree of success, which raise their self-confidence. Moreover, there are proposals that benefit from more challenging tasks (problems and investigations), which enable an effective mathematical experience to the students.

Exponential functions are models applied to analyze and describe a series of physical, chemical and economic phenomena and that is one of the main factors that underlie the epistemological relevance of the content which is emphasized in the BNCC (Brasil, 2017) and, consequently, in the textbooks. Therefore, in this work, we focus on two task proposals, thinking about how to introduce the content (Proposal 1) and how to consolidate it (Proposal 2). We hope, therefore, that based on the proposition presented, when developing it in real classroom situations, the teacher will have guidance on how to explore all the potential of the content, taking into account the nature of the aforementioned tasks, their registers of semiotic representation and the GeoGebra tool.

The GeoGebra allows simultaneous work with tabular, algebraic and graphical representations of those functions, besides enabling their relationship with the context in which the content is discussed, that is, its representation in natural language. Duval (2012b), in fact, confirms that such representation should not be neglected in the teaching of Mathematics in Basic Education, as it is as essential as other types, particularly those in which treatments and calculations are possible.

Being able to understand the correlation that exists, mainly among algebraic expression, graph and context (simultaneously), is one of the aspects that justify the importance of the GeoGebra when allowing students to recognize, here, the exponential function and functions of the exponential type in more than one register. In this sense, you will be able to perceive the relationship that exists, for example, among an algebraic coefficient, the geometric slope and the speed with which the temperature of a given liquid heats or cools, up to its ambient temperature, which, in turn, constitutes a distinctive significant unit that in the algebraic expression appears as a constant that adds to an exponential type function $f(x) = b \cdot a^{c \cdot x}$ and that in the Cartesian plane is a horizontal asymptote.

The teacher might then mediate the exploration of a series of concepts, at the same time, making connections between registers of semiotic representation with the students, being able to study and analyze what the variation of a significant unit in one changes and influences in the other (Duval, 2012a, 2012b, 2017). The experimental character that the GeoGebra attributes to the class is, in this way, an important teaching element, which allows the class to develop their autonomy and to experience and simulate exploratory mathematical experiences centered on conjectures, tests and discoveries. Such an action will be a counterpoint to the direct teaching discussed by Ponte (2005), in which the information is given ready-made and has the student's passivity as a striking characteristic.

Thus, exploratory teaching benefits from the articulation of registers of exponential functions with the support of the GeoGebra, by which means it is possible to instigate interesting discussions among teacher and students, without them being tied to directed answers or to objective questions from the teacher. Proposals such as number 2 allow that type of situation, by placing students as active subjects in the process of mathematizing a natural behavior, by allowing them to discover, on their own, certain properties and, mainly, to come across obstacles that can be excellent teaching motivators.

We confirm the importance of the teachers' stance in that type of intervention, which should not be sporadic or restricted to certain methods. The fact that they are not the only central figure who conveys the class and proposes questions does not limit their action, but rather expands their possibilities of interaction with the class and their approach to the subject. It is clear here that such an exploratory methodology does not exclude the application of closed structure tasks, much less the need for an expository class.

At various times in both proposals, gaps arise that require both a good theoretical explanation of the subject — asymptote, base changes, historical aspects, properties and basic demonstrations about powers and logarithms — as well as tasks that, before or after each proposal, aim to understand and to practice some concepts (exercises and problems). It's all a matter of diversifying and of knowing how to apply each type of task at the right time, how much to use and critically explore the textbook, in addition to identifying types of proposals that benefit the study of a given content. Remembering, also, that, even in the case of an open-structured task, it is essential that the teacher maintains a certain degree of leadership in the class, given that it is easy for students to get lost in daydreams and not be able to reach the desired skills, as a result of the very *free* aspect of the class.

This research may inspire other researchers and teachers to carry out studies of other mathematical objects, either with the same approach or by means of other exploratory methods involving functions with exponential behavior, as they model a series of other phenomena that can be replicated or analyzed in the classroom.

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